To the memory of Arthur Prior

Formal properties of 'now'

by

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This paper is about the word 'now'. It is closely related to the article [2] by professor A. N. Prior. In that article Prior gives an extensive, and undoudbtedly correct analysis of the semantical function of the word 'now' in ordinary discourse. He then develops a number of logical calculi which contain formal counterparts of the word 'now', as well as of certain other temporal notions and the truth-functional connectives. In some of these calculi 'now' is formalized as a l-place propositional connective, while in others it is represented by a propositional constant. As I believe that 'now'—like so many adverbs and adverbial clauses—should be regarded as a propositional modifier, I prefer the calculi of the first sort; and thus I will restrict my attention to them.

I will for these, and similar, calculi formulate the semantics, and then prove a number of metamathematical results about them. The most important of these results have the following form: Let $\mathcal{C}(\mathsf{N})$ be such a calculus, and let $\mathcal{C}$ be the calculus obtained by omitting the 'now'-operator from $\mathcal{C}(\mathsf{N})$. If an axiom system $\mathcal{A}$ for $\mathcal{C}$ is semantically complete then so is a closely to $\mathcal{A}$ related axiom system $\mathcal{A}'$ for $\mathcal{C}(\mathsf{N})$.

These results provide us with a number of different completeness theorems for some of the 'now'-calculi considered, since for the corresponding calculi without the 'now'-operator, many complete axiom systems are already available.

For propositional calculi these results are quite easily obtained. For in these calculi the 'now'-operator is, as it will turn out, always eliminable—i.e., every formula containing the operator is equivalent to a formula in which the operator does not occur. Thus, to obtain from a given axiom system $\mathcal{A}$ for the calculus without 'now' a complete axiom system for the calculus with 'now' it suffices to add axioms which make it possible to prove the equivalence between any formula and its 'now'-free equivalent.

In the predicate calculi which we will consider a formula containing 'now' is not always equivalent to a formula without 'now'. This fact is intuitively obvious, but nonetheless somewhat difficult to show. A proof is given in the last section. Because not all formulae of our predicate calculi are equivalent to 'now'-free formulae the completeness result has to be proved by means other than those used for the propositional calculi. The results obtained are somewhat less general than those for the propositional cases.

Before I proceed with the formal part of this paper I will, on the danger of repeating some of the points of Prior's article, give a short informal discussion of the behaviour of 'now' in English. My reason for this is twofold. In the first place I hope that a few informal remarks will make it easier to understand the formal definitions of the semantics which will follow later. In the second place I want to make it clear—before I embark upon technical developments which otherwise might seem pointless—that the word 'now' is not vacuous, in the sense that whenever someone makes a true, or false, statement by uttering a certain sentence in which the word 'now' occurs, he would also have made a true, or false, statement if he had uttered instead the sentence which is obtained if the word 'now' is omitted from the first sentence.

Some people have indeed thought that 'now' is vacuous in this sense. If they had been right this paper should not have been written. But they are not. To see this let us consider an argument which is sometimes given in support of the view that 'now' is vacuous. It starts from the following observation: Suppose that I make a true statement by uttering at time t a certain sentence, e.g., the sentence 'it is raining'. Then I would also have made a true statement if I had uttered at t the words 'it is now raining'. Similarly, if the statement made by uttering the first sentence had been false, then so would have been the statement made by uttering the second sentence. This observation is certainly correct. And it remains correct if we replace the words 'it is raining' by any other English sentence in the present tense. But it is wrong to conclude from this that all occurrences of the word 'now' are vacuous. In fact, consider the sentences:

(1) 'I learned last week that there would be an earthquake.'
(2) 'I learned last week that there would now be an earthquake.'

Obviously there could be circumstances under which I would make a true statement if I uttered the first sentence, but a false one if I uttered the second.

The function of the word 'now' in (2) is clearly to make the clause to which it applies—i.e., 'there would be an earthquake'—refer to the moment of utterance of (2), and *not* to the moment, or moments, (indicated by other temporal modifiers that occur in the sentence) to which the clause would refer (as it does in (1)) if the word 'now' were absent. A little reflection shows that this principle correctly describes the function of the word 'now' in all of its occurrences. It explains in particular why the occurrence of 'now' in 'it is now raining' is vacuous. For there the clause to which 'now' applies, viz., 'it is raining', is understood in any case to refer to the moment of utterance, whether 'now' be present or not.

This establishes that the word 'now' does not always occur vacuously. However, we have also seen that an occurrence of 'now' can be only non-vacuous if it occurs within the scope of another temporal modifier. Thus a formal analysis of 'now' will be of any interest only if it takes also other temporal operators into account. As a matter of fact the most interesting non-vacuous occurrences of 'now' are in sentences which contain besides such other temporal operators also propositional modifiers of a non-temporal character, e.g., modal, epistemic, or deontic operators. Sentence (1), in which the operator 'I learn that' occurs, is a case in point.

In this paper I will nonetheless consider besides 'now' only operators of a purely temporal nature. The reason is mainly one of expedience: For those operators which I will consider a comprehensive and satisfactory analysis has already been carried out, and we know a great deal about the systems to which this analysis has led. I will make use of that information to obtain the formal results which will follow. On the other hand relatively few formal systems are so far available in which temporal as well as non-temporal operators are represented. I believe, however, that the analysis given here will make the extension of such systems with a 'now'-operator a straightforward matter once they will have been developed without 'now'.

My brief statement of the general function of the word 'now' may suggest that, even if the word is not vacuous, its semantical behaviour is too simple to justify a formal analysis. However, it turns out that a proper treatment of 'now' together with the other temporal operators which I will consider is not totally trivial, as the semantics developed earlier for these operators can not be extended in an entirely automatic manner so as to cover 'now'.

Nonetheless the reader may still have the feeling that the amount of attention paid to the word 'now' in this paper is excessive. I will end this introduction with a few remarks aimed at dispelling that feeling.

In the first place it should be observed that the feature which distinguishes 'now' from those temporal modifiers which had already been satisfactorily treated previously is to be found also in a number of other temporal concepts, e.g., in those expressed by the words 'today', 'yesterday', 'last week', 'next year', etc., as well as, to some extent, in the word 'then'. Thus the analysis of 'now' given here is also a paradigm for similar analyses of those other concepts.

In the second place I want to point at a phenomenon which is connected with the use in English of the ordinary past and future tenses, but which, to my knowledge, has so far been overlooked by tense logicians, and which is intimately related to the analysis of 'now' given here. I have said repeatedly in this introduction that a satisfactory analysis already exists for a number of temporal notions. Among these notions are the past and future tenses. The past tense is in this analysis represented by a 1-place propositional operator P, which should be thought of as transforming, in particular, sentences in the present tense into the corresponding sentences in the past tense. Thus $P\varphi$ can be read as 'it was the case that φ', and, in particular, if φ is, e.g., the sentence 'it rains', as 'it rained'. Similarly the future tense is represented by the 1-place operator F, so that $F\varphi$ can be read as 'it will be the case that φ'. The semantics stipulates that a formula $P\varphi$ is true at a moment t if and only if φ is true at some moment preceding t; and that $F\varphi$ is true at t if and only if φ is true at some moment following t. Thus, in particular, $PF\varphi$ is true at t if and only if there is a moment t' before t such that φ is true at some moment later than t'. Now compare the following two sentences:

(3) 'A child was born that would become ruler of the world.'
(4) 'A child was born that will become ruler of the world.'

It is clear that while the first sentence would be true if at some past time t a child was born to become ruler of the world at some time t' later than t—whether that time t' be before, identical with, or later than the present—the second sentence would be true *only* if the child is to become ruler at a time later than the present. It follows that (3) can be correctly rendered in the following form:

$$P(\exists x)\,(x \text{ is born} \wedge F(x \text{ is ruler of the world})).$$

But for (4) no correct symbolization with the help of only P, F, and the apparatus of ordinary predicate logic is possible.[1] However, if we have at our disposal also the 'now'-operator N, (which will be introduced in Section 2), we can symbolize (4) properly as

[1] I of course exclude the possibility of symbolizing the sentence by means of explicit quantification over moments. Such a symbolization of (2) would certainly be possible; and it would even make the operators P and F superfluous. Such symbolizations, however, are a considerable departure from the actual form of the original sentences which they represent—which is unsatisfactory if we want to gain insight into the semantics of English. Moreover, one can object to symbolizations involving quantification over such abstract objects as moments, if these objects are not explicitly mentioned in the sentences that are to be symbolized.

$\mathsf{P}(\exists x)(x$ is born $\wedge$ $\mathsf{NF}(x$ is ruler of the world$))$.

The semantics developed in Section 2 will indeed show that this symbolization is adequate.

§ 1.

I will assume throughout this paper the existence of an infinite class $\mathcal{E}$ and a well-founded concatenation function C on $\mathcal{E}$.[2] $\mathcal{E}$ will be referred to as the *set of expressions*. S will be the class $\mathcal{E}$-Range C; its members will be referred to as the *symbols* of $\mathcal{E}$. Whenever $e_1, e_2 \in \mathcal{E}$, we will write e_1e_2 instead of $\mathsf{C}(\langle e_1, e_2\rangle)$.

We assume that S contains symbols (,) (called *parentheses*); $q_1, q_2, \ldots$ (called *propositional constants*); and for each $n \in \omega$ symbols $\mathsf{C}_0^n, \mathsf{C}_1^n, \mathsf{C}_2^n, \ldots$ (called *n-place connectives*). We will refer to $\mathsf{C}_0^1, \mathsf{C}_1^1, \mathsf{C}_2^1, \mathsf{C}_3^1, \mathsf{C}_4^2$ as N, $\sim$, G, H, $\wedge$, respectively.

By a *language for propositional tense logic* we understand a set consisting of the symbols (,), q_i $(i = 1, 2, \ldots)$ and some of the symbols C_i^n. The *formulae* of a language $\mathcal{L}$ for propositional tense logic are defined by:

DEFINITION 1. (i) q_i is a *formula* of $\mathcal{L}$; (ii) if $\mathsf{C}_i^n \in \mathcal{L}$ and $\varphi_1, \ldots, \varphi_n$ are formulae of $\mathcal{L}$ then $\mathsf{C}_i^n(\varphi_1 \ldots \varphi_n)$ is a *formula* of $\mathcal{L}$.

I will always write $(\varphi\wedge\psi)$ in stead of $\wedge(\varphi\ \psi)$. Furthermore, $(\varphi\vee\psi)$ will stand for $\sim(\sim\varphi \wedge \sim\psi)$; $(\varphi \rightarrow \psi)$ will stand for $(\sim\varphi \vee \psi)$; $(\varphi \leftrightarrow \psi)$ will stand for $(\varphi \rightarrow \psi)\wedge(\psi \rightarrow \varphi)$; $\mathsf{P}\varphi$ for $\sim(\mathsf{H}(\sim)\varphi)))$; and $\mathsf{F}\varphi$ for $\sim(\mathsf{G}(\sim(\varphi)))$. Parentheses will be omitted whenever no confusion is possible. In particular I will always write $\sim\varphi$, $\mathsf{N}\varphi$,

[2] By a *well-founded concatenation function* $\mathfrak{J}$ on a class S we understand a function from $S \times S$ into S, such that:

(i) $\mathfrak{J}(\langle\mathfrak{J}(\langle s_1, s_2\rangle), s_3\rangle) = \mathfrak{J}(\langle s_1, \mathfrak{J}(\langle s_2, s_3\rangle)\rangle)$;

(ii) if $\mathfrak{J}(\langle s_1, s_2\rangle) = \mathfrak{J}(\langle s_3, s_4\rangle)$ then either $(s_1 = s_3$ and $s_2 = s_4)$ or there is an $s \in S$ such that $s_1 = \mathfrak{J}(\langle s_3, s\rangle)$ and $s_4 = \mathfrak{J}(\langle s, s_2\rangle)$ or there is an $s \in S$ such that $s_3 = \mathfrak{J}(\langle s_1, s\rangle)$ and $s_2 = \mathfrak{J}(\langle s, s_4\rangle)$;

(iii) there is no infinite sequence $s_0, s_1, s_2, \ldots$ of elements of S such that for each $n \in \omega$ there is a $t \in S$ such that $s_n = \mathfrak{J}(t, s_{n+1})$ or $s_n = \mathfrak{J}(s_{n+1}, t)$.

Hφ, Gφ instead of $\sim(\varphi)$, N(φ), H(φ), G(φ), respectively. The formulae Nφ, Hφ, Gφ are read as 'it is now the case that φ', 'it has always been the case that φ', and 'it will always be the case that φ', respectively.

T will always be a non-empty set and $<$ a partial ordering (i. e., a transitive and asymmetric relation) on T. $\mathcal{T}$ will be the pair $<T, <>$. We think of T as the set of moments and of $<$ as the earlier-later relation between them.[3] The numbers 0 and 1 will be used as truth-values, 1 for truth and 0 for falsehood.

Before turning to the 'now'-calculus itself, I will first consider a simpler system, $\mathcal{L}_1$, the connectives of which are $\sim$, $\wedge$, H, G. As the syntax of $\mathcal{L}_1$ has already been defined, I will proceed at once with the semantics.

I feel that this paper—which is concerned with the specific properties of 'now'—is not the place where I should give an extensive justification of the principles on which tense logic is based and which underlie in particular its semantics. Therefore I will restrict myself to a few explanatory remarks to make the following formal definitions more understandable.

Tense logic belongs to a type of logic in which is explicitly adopted a feature of natural languages that has no part in the standard systems of mathematical logic (as, e.g., the ordinary predicate calculus): In natural languages we can in many cases use the same linguistic form under different circumstances—and, in particular, at different times—to make different statements. So we may use the expression 'it rains' at one time to make a true statement and at another to make a false one. Since the truth-value of the statement made by using the expression

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[3] Unlike Prior I assume, throughout this paper, that the relation 'earlier than' between moments of time is a partial ordering. My reason is this: Tense logic is concerned with the analysis of the logical properties of certain temporal notions. Such an analysis must perforce start from the naïve intuitions which we have about these notions. Now, in my opinion, our belief that the earlier-later relation between moments is a partial ordering plays such a central role in the network of intuitions which we have about temporal notions, that unless we uphold that belief we have no ground for relying, to any degree whatever, on our intuitions about the notions which we want to analyze.

at t varies with t, the expression can not be said to have a truth-value all by itself. However, if we assume that the truth-value of the statement is completely determined by (a) the expression used and (b) the time of utterance,[4] we may conclude that the truth-value of the statement made can be regarded as a function of only these two factors; and thus we may consider for every (appropriate) expression the concept of 'the *truth-value of* that expression *at* an arbitrary moment t'. With the help of this concept of 'being true at t' we can give a semantical analysis of those temporal notions with which tense logic is concerned, e.g., the notion represented in $\mathfrak{L}_1$ by the operator H. For we can state in a general and systematic way how the truth-value of a formula $H\varphi$ depends on the truth-values of φ at t *and at other moments*. For this purpose we need semantic structures which are not the models of standard model theory—which simply specify a truth-value for each atomic formula of the language that they interpret—but rather structures which specify for every formula a truth-value at each moment of time. This is just what the interpretations defined below do.

DEFINITION 2. $\mathcal{M}$ is an *interpretation for* $\mathfrak{L}_1$ *relative to* $\mathcal{T}$, iff $\mathcal{M}$ is a function such that

(1) the domain of $\mathcal{M}$ consists of $q_1, q_2, \ldots$; and
(2) for each $i > 0$ $\mathcal{M}(q_i)$ is a function with domain $\mathcal{T}$ and range included in $\{0, 1\}$.

On the basis of the preceding remarks the truth definition for complex formulae is straightforward given the intended meanings of the connectives $\sim$, $\wedge$, H, G.

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[4] This assumption is of course an idealization, which precludes the proper treatment of large portions of natural languages, viz., of those parts where the truth-value of the statement made depends not only on the form of the expression used and the time of utterance, but also on other aspects of the situation, e.g., the identity of the speaker—compare, e.g., the sentence of the words 'I am hungry'. However, to the analysis of the purely temporal notions with which tense logic is concerned, this idealization will do no harm.

DEFINITION 3. Let $\mathcal{M}$ be an interpretation for $\mathcal{L}_1$, relative to $\mathcal{T}$. For any formula φ of $\mathcal{L}_1$ and $t \in T$ the *truth-value of* φ *in* $\mathcal{M}$ *at* t, *relative to* $\mathcal{T}$, (in symbols: $[\varphi]^{\mathcal{T}}_{\mathcal{M},t}$) is defined as follows:

(1) $[q_i]^{\mathcal{T}}_{\mathcal{M},t} = 1$ iff $\mathcal{M}(q_i) = 1$;
(2) if φ, ψ are formulae of $\mathcal{L}_1$, then

$$\text{(i)}\quad [\sim\varphi]^{\mathcal{T}}_{\mathcal{M},t} = \begin{cases} 1 \text{ if } [\varphi]^{\mathcal{T}}_{\mathcal{M},t} = 0; \\ 0 \text{ otherwise;} \end{cases}$$

$$\text{(ii)}\quad [\varphi \wedge \psi]^{\mathcal{T}}_{\mathcal{M},t} = \begin{cases} 1 \text{ if } [\varphi]^{\mathcal{T}}_{\mathcal{M},t} = 1 \text{ and } [\psi]^{\mathcal{T}}_{\mathcal{M},t} = 1; \\ 0 \text{ otherwise;} \end{cases}$$

$$\text{(iii)}\quad [\mathsf{H}\varphi]^{\mathcal{T}}_{\mathcal{M},t} = \begin{cases} 1 \text{ for if all } t' \in T \text{ such that } t' < t\ [\varphi]^{\mathcal{T}}_{\mathcal{M},t'} = 1; \\ 0 \text{ otherwise;} \end{cases}$$

$$\text{(iv)}\quad [\mathsf{G}\varphi]^{\mathcal{T}}_{\mathcal{M},t} = \begin{cases} 1 \text{ if for all } t' \in T \text{ such that } t < t'\ [\varphi]^{\mathcal{T}}_{\mathcal{M},t'} = 1; \\ 0 \text{ otherwise.} \end{cases}$$

DEFINITION 4. A formula φ of $\mathcal{L}_1$ is *valid, relative to* $\mathcal{T}$, iff for every interpretation $\mathcal{M}$ for $\mathcal{L}_1$, relative to $\mathcal{T}$, and every $t \in T$, $[\varphi]^{\mathcal{T}}_{\mathcal{M},t} = 1$.

Clauses 2. (iii) and 2. (iv) of definition 3 suggest that the set of valid formulae of $\mathcal{L}_1$ may depend on the structure of $\mathcal{T}$. This is indeed the case: There are partial orderings $\mathcal{T}$ and $\mathcal{T}'$ such that the set of valid formulae, relative to $\mathcal{T}$, is different from the set of valid formulae, relative to $\mathcal{T}'$. Thus, intuitively, we may, in as far as we are ignorant of the structure of time, well be unable to determine which formulae are to be regarded as intuitively 'tense logically valid' (i.e., valid on the basis only of how propositional connectivas and tenses occur in them). However, once we assume that time has *certain* properties (e.g., that it is dense) we can regard at least some formulae as tense-logically valid, viz., those which are valid relative to all partial orderings which have these properties. It is thus natural to introduce the following notion of validity.

DEFINITION 5. Let $\mathcal{K}$ be a non-empty class of partial orderings. A formula φ of $\mathcal{L}_1$ in *$\mathcal{K}$-valid* iff for every $\mathcal{T} \in \mathcal{K}$, φ is valid, relative to $\mathcal{T}$.

There is a slightly different, but obviously equivalent, way in which we can develop the semantics for $\mathcal{L}_1$. This development makes use of a new kind of interpretation. In order to avoid ambiguity I will call interpretations of this new kind *interpretations$_2$*, and I will refer to the interpretations of Definition 3 as *interpretations$_1$*.

DEFINITION 6. An *interpretation$_2$ for $\mathcal{L}_1$, relative to $\mathcal{T}$*, is an ordered pair consisting of an interpretation$_1$ for $\mathcal{L}_1$, relative to $\mathcal{T}$, and a member of T.

DEFINITION 7. For every interpretation$_2$ $<\mathcal{M}, t_0>$ for $\mathcal{L}_1$, relative to $\mathcal{T}$, any formula φ of $\mathcal{L}_1$ and any $t \in T$, the *truth-value of φ in $<\mathcal{M}, t_0>$ at t, relative to $\mathcal{T}$* (in symbols $[\varphi]^{\mathcal{T}}_{<\mathcal{M}, t_0>, t}$) is the truth-value of φ in $\mathcal{M}$ at t, relative to $\mathcal{T}$.

DEFINITION 8. φ is *valid$_2$, relative to $\mathcal{T}$*, if for every interpretation$_2$ $<\mathcal{M}, t_0>$ for $\mathcal{L}_1$, relative to $\mathcal{T}$, $[\varphi]^{\mathcal{T}}_{<\mathcal{M}, t_0>, t_0} = 1$.

As can be seen from Definitions 2, 3, 4, 6, 7, 8 the two developments differ only in their respective characterizations of validity in terms of truth. While according to the first characterization (Definition 4) a formula is valid only if it is true in each interpretation at each moment, the second characterization (Definition 8) demands only that the formula be true in each interpretation at one particular moment. One may think of that moment as the 'present' of the interpretation in question. Thus Definition 4 is based on the idea that we are, so to speak, interested only in what formulae are true at the present time; the truth-values of formulae at other moments are important only in so far as they determine the truth-values of certain more complex formulae at this present time.

At this point the two approaches are of course trivially equivalent. But we will see later that when we add the operator N to $\mathcal{L}_1$, to represent 'now', the difference between the two approaches becomes important; and that while the first is probably intuitively the more natural, the second leads to a considerable technical simplification of which we will make use in subsequent proofs.

Finding axion systems which generate the 'valid' formulae of $\mathcal{L}_1$ is a complex task. Indeed, every particular partial ordering $\mathcal{T}$ gives rise to an axiomatization problem of its own—viz., the problem of finding an axiom-system that will generate all and only those formulae which are valid relative to $\mathcal{T}$—and so does every non-empty class of partial orderings. Problems of this sort, however, will not be of my concern in this paper. Rather I will show how an axiom system $\mathcal{A}$ for $\mathcal{L}_1$ can be modified into an axiom system $\mathcal{A}'$ for the 'now'-calculus defined below, so that if $\mathcal{A}$ generates the set of all formulae of $\mathcal{L}_1$ which are valid relative to $\mathcal{T}$ [$\mathcal{K}$-valid] then $\mathcal{A}'$ will generate the set of all formulae of the 'now'-calculus which are valid relative to $\mathcal{T}$ [$\mathcal{K}$-valid]. Since for several partial orderings and classes of partial orderings axiom systems generating the corresponding sets of valid formulas have already been given by others, this procedure will provide us with an equal number of complete axiom systems for the 'now'-calculus.

We now turn to the 'now'-calculus itself. Let $\mathcal{L}_2$ be the language $\mathcal{L}_1 \cup \{\mathsf{N}\}$. The *interpretations for $\mathcal{L}_2$, relative to $\mathcal{T}$,* are simply the interpretations for $\mathcal{L}_1$, relative to $\mathcal{T}$.

The truth definition for $\mathcal{L}_2$, however, cannot be obtained by a straightforward adaptation of the corresponding definition for $\mathcal{L}_1$ (Definition 3). The difficulty stems from the peculiar behavior of the word 'now', which our truth definition should reflect. An essential feature of the word 'now' is that it always refers back to the moment of utterance of the sentence in which it occurs, even if it stands itself in that sentence within the scope of one or more tenses. It is this feature that makes the English counterparts of, e.g., $\varphi \to \mathsf{N}\varphi \wedge \mathsf{HN}\varphi \wedge \mathsf{GN}\varphi$ ('if it is the case that φ, then it is now the case that φ, it always has been the case that it is now the case that φ, and it always will be the case that it is now the case that φ') logically true. Thus our truth definition should be such that it makes in particular this formula valid. Such a truth definition cannot be obtained simply by adding to Definition 3 a clause of the form:

$$(1)\quad [\mathsf{N}\varphi]_{\mathfrak{M},\,t} = \begin{cases} 1 \text{ iff } \Phi; \\ 0 \text{ otherwise.} \end{cases}$$

For if the definition correctly reflects the behavior of 'now' then it should make the formula

$$(2)\quad q_0 \leftrightarrow \mathsf{N}q_0$$

valid, relative to any partial ordering, as any English counterpart of this formula is clearly true irrespective of the structure of time. This implies that the condition Φ in (1) should be equivalent to the condition that $[\varphi]^{\mathcal{T}}_{\mathcal{M},t} = 1$. But if Φ is equivalent to this condition, then the formula $q_1 \rightarrow \mathsf{N}q_1 \wedge \mathsf{HN}q_1 \wedge \mathsf{GN}q_1$ will be not valid, relative to any partial ordering $\mathcal{T}$ which has at least two points. For in any interpretation based upon such a $\mathcal{T}$ in which q_1 is true at only one moment, t say, $\mathsf{N}q_1$ will, according to (1), also be true only at t, and therefore $q_1 \rightarrow \mathsf{N}q_1 \wedge \mathsf{HN}q_1 \wedge \mathsf{GN}q_1$ will be false at t. Thus an adequate truth definition of this form cannot be found.

This argument may suggest where the difficulty lies: the truth-values of $\mathsf{HN}q_1$ ($\mathsf{GN}q_1$) depend on the truth values of $\mathsf{N}q_1$ in a way which is different from the manner in which the truth values of $\mathsf{H}q_1$ ($\mathsf{G}q_1$) depend on the truth values of q_1; and any definition obtained by adding a clause of the form (1) to Definition 3 will be incapable of doing justice to that difference. In order to find an appropriate truth definition let us recall the remark, made above, that "the word 'now' refers back to the moment of utterance". In view of this fact we should, if we want to analyse the truth of formulae that contain N in terms of the truth-values of their components, "keep track" during this analysis of the moment of utterance of the entire expression. The concept we ought to analyze is not simply "the truth-value of φ at t", but rather "the truth-values of φ at t when part of an utterance made at t'". Of course, our real interest is in the truth-value of a sentence at the moment of *its* utterance. But the analysis of this truth-value in terms of the truth-values of the components of the sentence will automatically lead to the consideration of truth-values of formulas at moments different from the moment of their utterance. We thus come to the following definition:

DEFINITION 9. Let $\mathcal{T}$ be a linear ordering. Let $\mathcal{M}$ be an interpretation of $\mathfrak{L}_2$, relative to $\mathcal{T}$. For any formula φ of $\mathfrak{L}_2$, and $t, t' \in T$, *the truth-value of* φ *in* $\mathcal{M}$ *at t when part of an expression uttered at* t', *relative to* $\mathcal{T}$ (in symbols: $[\varphi]^{\mathcal{T}}_{\mathcal{M},t,t'}$) is defined as follows:

(1) $[q_i]^{\mathcal{T}}_{\mathcal{M},t,t'} = \mathcal{M}(q_i)(t)$;

(2) if φ, ψ are formulas of $\mathcal{L}_2$, then

(i) $[\sim\varphi]^{\mathcal{T}}_{\mathcal{M},t,t'} = \begin{cases} 1 \text{ if } [\varphi]^{\mathcal{T}}_{\mathcal{M},t,t'} = 0; \\ 0 \text{ otherwise}; \end{cases}$

(ii) $[(\varphi \wedge \psi)]^{\mathcal{T}}_{\mathcal{M},t,t'} = \begin{cases} 1 \text{ if } [\varphi]^{\mathcal{T}}_{\mathcal{M},t,t'} = 1 \text{ and} \\ \quad [\psi]^{\mathcal{T}}_{\mathcal{M},t,t'} = 1; \\ 0 \text{ otherwise}; \end{cases}$

(iii) $[\mathsf{H}\varphi]^{\mathcal{T}}_{\mathcal{M},t,t'} = \begin{cases} 1 \text{ if for all } t'' \in T, \text{ such that} \\ \quad t'' < t,\ [\varphi]^{\mathcal{T}}_{\mathcal{M},t'',t'} = 1; \\ 0 \text{ otherwise}; \end{cases}$

(iv) $[\mathsf{G}\varphi]^{\mathcal{T}}_{\mathcal{M},t,t'} = \begin{cases} 1 \text{ if for all } t'' \in T \text{ such that} \\ \quad t < t'',\ [\varphi]^{\mathcal{T}}_{\mathcal{M},t'',t'} = 1; \\ 0 \text{ otherwise}; \end{cases}$

(v) $[\mathsf{N}\varphi]^{\mathcal{T}}_{\mathcal{M},t,t'} = \begin{cases} 1 \text{ if } [\varphi]^{\mathcal{T}}_{\mathcal{M},t',t'} = 1; \\ 0 \text{ otherwise}; \end{cases}$

A formula should be regarded as valid if in every interpretation it is *true when uttered*. Thus validity should be defined as follows:

DEFINITION 10. A formula φ of $\mathcal{L}_2$ is *valid, relative to* $\mathcal{T}$, iff for every interpretation $\mathcal{M}$ for $\mathcal{L}_2$ relative to $\mathcal{T}$, and every $t \in T$, $[\varphi]^{\mathcal{T}}_{\mathcal{M},t,t} = 1$.

Since $\mathcal{L}_1$ and $\mathcal{L}_2$ have many formulae in common, it is conceivable that Definitions 9 and 10 clash with Definitions 3 and 4; a formula of both $\mathcal{L}_1$ and $\mathcal{L}_2$ could be valid, relative to $\mathcal{T}$, according to Definition 4, but not valid according to Definition 12; or vice versa. Such a clash would of course imply that at least one of the definitions is inappropriate. However, this is not the case. One can easily show (we omit the proof) that if φ is a formula of both $\mathcal{L}_1$ and $\mathcal{L}_2$ then φ is valid, relative to $\mathcal{T}$, according to Definition 4, if and only if φ is valid, relative to $\mathcal{T}$, according to Definition 10.

We will now consider the analogues for $\mathcal{L}_2$ of the interpretations for $\mathcal{L}_1$.

DEFINITION 11. An *interpretation*$_2$ for $\mathcal{L}_2$, *relative to* $\mathcal{J}$, is a pair consisting of an interpretation$_1$ for $\mathcal{L}_2$, relative to $\mathcal{J}$, and a member of T.

As I said before when considering interpretations we are primarily interested in the truth-values of formulae at only one particular moment. Therefore we can avoid the complications that arose in connection with the truth definition for $\mathcal{L}_2$ applicable to interpretations$_1$. Indeed, the truth definition can be given as a simple extension of Definition 7.

DEFINITION 12. Let $\mathcal{M} = \langle \mathcal{M}', t_0 \rangle$ be an interpretation$_2$ for $\mathcal{L}_2$, relative to $\mathcal{J}$. For any formula φ of $\mathcal{L}_2$ and $t \in T$, the *truth-value of* φ *in* $\mathcal{M}$ *at* t, *relative to* $\mathcal{J}$ (in symbols: $[\varphi]^{\mathcal{J}}_{\mathcal{M},t}$) is defined as follows:

(1), (2) (i)-(iv): as in Definition 3;

(2) (v) $[N\varphi]^{\mathcal{J}}_{\mathcal{M},t} = \begin{cases} 1 \text{ if } [\varphi]^{\mathcal{J}}_{\mathcal{M},t_0} = 1; \\ 0 \text{ otherwise.} \end{cases}$

As before validity$_2$ is defined by:

DEFINITION 13. A formula φ of $\mathcal{L}_2$ is *valid*$_2$ if for every interpretation$_2$ $\langle \mathcal{M}, t \rangle$ relative to $\mathcal{J}$, $[\varphi]_{\langle \mathcal{M},t \rangle, t} = 1$.

It is obvious that this definition does not conflict with Definition 8: every formula of $\mathcal{L}_2$ which is also a formula of $\mathcal{L}_2$ is valid$_2$ in the sense of Definition 8 iff it is valid$_2$ in the sense of Definition 15. This follows from previous remarks, and the fact that every formula of $\mathcal{L}_2$ is valid$_2$, relative to $\mathcal{J}$, iff it is valid, relative to $\mathcal{J}$. The latter fact is true since for every interpretation$_1$ $\mathcal{M}$ for $\mathcal{L}_2$, relative to $\mathcal{J}$ and t_0, $t \in T$, and every formula φ of $\mathcal{L}_2$,

$$[\varphi]^{\mathcal{J}}_{\mathcal{M},t} = [\varphi]^{\mathcal{J}}_{\langle \mathcal{M},t_0 \rangle, t}$$

(which can be shown by an easy induction argument, omitted here).

In the proofs of the theorems below it will be somewhat more convenient to work with interpretations$_2$ than with interpretations$_1$. Therefore we now drop interpretations$_1$ altogether and will refer to interpretations$_2$ simply as *interpretations*. Also we will speak of *validity* instead of validity$_2$.

§ 2.

As I have said in the introduction, the results presented in this paper apply to arbitrary axiom systems. Also, I will later introduce unusual forms of proof from axiom systems. In view of these two facts it will be necessary to give a precise account of the general notion of an *axiom system* and of a *proof from* an axiom system which will be used in the sequel. Definition 15 provides this account. Definition 14, which precedes it, is concerned with the notion of substitution of formulae for propositional constants which is essential for that account.

DEFINITION 14. Let φ, $\psi_1, \ldots, \psi_k$ be formulae of some language $\mathfrak{L}$ for propositional tense logic, and let $i_1, \ldots, i_k$ be positive integers. By $[\varphi]\ \psi_1/q_{i_1}, \ldots, \psi_k/q_{i_k}$ we understand the result of replacing in φ q_{i_1} everywhere by $\psi_1, \ldots, q_{i_k}$ by ψ_k. $[\varphi]\ \psi_1/q_{i_1}, \ldots, \psi_k/q_{i_k}$ is called an *instance of* φ *in* $\mathfrak{L}$.

DEFINITION 15. (1) An *inference rule in* $\mathfrak{L}$ is a pair consisting of a finite set of formulae of $\mathfrak{L}$ and a formula of $\mathfrak{L}$. If $R = \langle \Sigma, \varphi \rangle$ is an inference rule in $\mathfrak{L}$, we call the members of Σ the *premisses* of R and φ the *conclusion* of R. In case Σ is empty, R is called an *axiom* (*in* $\mathfrak{L}$).
(2) An *axiom system for* $\mathfrak{L}$ is a set of inference rules in $\mathfrak{L}$.
(3) Let $R = \langle \Sigma, \varphi \rangle$ be an inference rule in $\mathfrak{L}$. Let Γ be a set of formulae of $\mathfrak{L}$, ψ a formula of $\mathfrak{L}$. We say that ψ *follows from* Γ *by* R iff there are propositional constants $q_1, \ldots, q_k$, and formulae $\psi_1, \ldots, \psi_k$ of $\mathfrak{L}$, such that $\psi = [\ldots [[\varphi]\ \psi_1/q_1] \ldots]\ \psi_k/q_k$, and for each φ' in Σ $[\ldots [[\varphi']\ \psi_1/q_1] \ldots]\ \psi_k/q_k$ belongs to Γ.
(4) Let $\mathcal{A}$ be an axiom system for $\mathfrak{L}$. A *proof from* $\mathcal{A}$ *in* $\mathfrak{L}$ is a finite sequence of formulae of $\mathfrak{L}$ such that each member of the sequence follows from the preceding members in the sequence by one of the inference rules in $\mathcal{A}$. A formula of $\mathfrak{L}$ is said to be *provable from* $\mathcal{A}$ *in* $\mathfrak{L}$ iff it occurs in some proof from $\mathcal{A}$ in $\mathfrak{L}$.

If $\mathcal{A}$ is an axiom system for $\mathfrak{L}$ and φ is a formula of $\mathfrak{L}$ which is provable from $\mathcal{A}$ in some other language $\mathfrak{L}'$, then φ is provable from $\mathcal{A}$ in $\mathfrak{L}$. I will therefore omit reference to $\mathfrak{L}$ when speaking of provability.

For the remainder of this section we will limit our attention to axiom systems for $\mathcal{L}_1$ and $\mathcal{L}_2$. The characterization of the notions of *$\mathcal{K}$-consistency* and *$\mathcal{K}$-completeness* of an axiom system for $\mathcal{L}_1$ is almost straightforward. However, it is worthwhile to note that the distinction between what is usually referred to as *strong completeness* and *weak completeness* is here more important than in connection with ordinary propositional (or even predicate) calculus, since for $\mathcal{L}_1$ and $\mathcal{L}_2$ these two notions diverge. An example to show that the notions do not coincide in the case of $\mathcal{L}_1$ will follow the next definition.

DEFINITION 16. Let $\mathcal{K}$ be a non-empty class of partial orderings, $\mathcal{A}$ an axiom system for $\mathcal{L}_1$. (1) $\mathcal{A}$ is *$\mathcal{K}$-consistent in $\mathcal{L}_1$* iff every formula of $\mathcal{L}_1$ which is provable from $\mathcal{A}$ is $\mathcal{K}$-valid. (2) $\mathcal{A}$ is *weakly $\mathcal{K}$-complete in $\mathcal{L}_1$* iff every formula of $\mathcal{L}_1$ which is $\mathcal{K}$-valid is provable from $\mathcal{A}$. (3) A set Δ of formulae of $\mathcal{L}_1$ is *consistent relative to $\mathcal{A}$* iff there are no number $n \geq 1$ and formulae $\varphi_1, \ldots, \varphi_n \in \Delta$ such that $\sim(\varphi_1 \wedge \ldots \wedge \varphi_n)$ is provable. (4) $\mathcal{A}$ is *strongly $\mathcal{K}$-complete in $\mathcal{L}_1$* iff (i) for every set Δ of formulae of $\mathcal{L}_1$ which is consistent relative to $\mathcal{A}$ there is a $\mathfrak{T} \in \mathcal{K}$ and interpretation $\langle \mathcal{M}, t_0 \rangle$ relative to $\mathfrak{T}$ such that for all $\varphi \in \Delta$ $[\varphi]^{\mathfrak{T}}_{<\mathcal{M}, t_0>, t_0} = 1$; and (ii) if any formula $\sim \sim\varphi$ of $\mathcal{L}_1$ is provable from $\mathcal{A}$ then so is φ.

One easily verifies that if $\mathcal{A}$ is strongly $\mathcal{K}$-complete then it is weakly $\mathcal{K}$-complete. The converse is not true, for we know of an axiom system for $\mathcal{L}_1$ which is $\{\mathfrak{J}\}$-consistent and weakly $\{\mathfrak{J}\}$-complete, where $\mathfrak{J} = \langle J, < \rangle$, J is the set of integers and $<$ their natural ordering (see [1]). That such an axiom system cannot be strongly $\{\mathfrak{J}\}$-complete follows from the fact that the notion of truth at the point 0 in an interpretation relative to $\mathfrak{J}$ is not compact.[5] That it is not follows from the fact that for the set $\Delta =$

[5] A notion of truth for a language $\mathcal{L}$ is *compact* if the following is the case: Let $\mathcal{K}$ be any set of models for $\mathcal{L}$ such that whenever Δ is a set of formulae of $\mathcal{L}$ and there is a model for $\mathcal{L}$ in which all members of Δ are true, then there is a model in $\mathcal{K}$ for which this is also the case. Let $\mathfrak{T}$ be the topology on $\mathcal{K}$ defined by the condition that $\mathcal{K}' \subseteq \mathcal{K}$ is *open* iff there is a set Δ of formulae of $\mathcal{L}$ such that $\mathcal{K}'$ is the set of models in $\mathcal{K}$ in which all members of Δ are true. Then $\langle \mathcal{K}, \mathfrak{T} \rangle$ is compact.

$\{Fq_1, FFq_1, FFFq_1, \ldots, FG \sim q_1\}$ there is no interpretation $\langle \mathcal{M}, 0\rangle$ relative to $\mathcal{J}$ such that for all $\varphi \in \Delta$, $[\varphi]^{\mathcal{J}}_{<\mathcal{M}, 0>, 0} = 1$, while for every finite subset Δ' of Δ there does exist an interpretation $\langle \mathcal{M}, 0\rangle$, relative to $\mathcal{J}$, such that $[\varphi]^{\mathcal{J}}_{<\mathcal{M}, 0>, 0} = 1$ for all $\varphi \in \Delta'$.

The next definition tells us how to extend an axiom system $\mathcal{A}$ for $\mathcal{L}_1$ to an axiomsystem $\mathcal{A}'$ for $\mathcal{L}_2$ so that, roughly speaking, $\mathcal{A}'$ will be $\mathcal{K}$-consistent and $\mathcal{K}$-complete for $\mathcal{L}_2$ if $\mathcal{A}$ is $\mathcal{K}$-consistent and $\mathcal{K}$-complete for $\mathcal{L}_1$.

DEFINITION 17. Let $\mathcal{A}$ be an axiomsystem for $\mathcal{L}_1$. Then $\mathcal{A}'$ is the axiomsystem $\mathcal{A} \cup \{\langle\{q_1, q_1 \rightarrow q_2\}, q_2\rangle, \langle\{q_1\}, Lq_1\rangle, \langle\emptyset, L(Nq_1 \rightarrow LNq_1)\rangle, \langle\emptyset, Lq_1 \rightarrow Nq_1\rangle, \langle\emptyset, \sim Nq_1 \leftrightarrow N \sim q_1\rangle, \langle\emptyset, N(q_1 \rightarrow q_2) \rightarrow (Nq_1 \rightarrow Nq_2)\rangle, \langle\emptyset, q_1 \leftrightarrow Nq_1\rangle\}$.[6]

N. B. we will refer to the axioms $Lq_1 \rightarrow Nq_1$, $\sim Nq_1 \leftrightarrow N \sim q_1$, $N(q_1 \rightarrow q_2) \rightarrow (Nq_1 \rightarrow Nq_2)$, $q_1 \leftrightarrow Nq_1$, $L(Nq_1 \rightarrow LNq_1)$ as N_L, $N_\sim$, $N_\rightarrow$, N_1 and N_2, respectively. The rule $\langle\{q_1, q_1 \rightarrow q_2\}, q_2\rangle$ is called M(odus) P(onens).

One easily verifies that if $\mathcal{K}$ is a non-empty class of partial orderings and $\mathcal{A}$ is an axiom system for $\mathcal{L}_1$ which is $\mathcal{K}$-consistent in $\mathcal{L}_1$, then $\mathcal{A}'$ will not necessarily be $\mathcal{K}$-consistent in $\mathcal{L}_2$. In fact, suppose that $\mathcal{K}$ contains linear orderings with more than one point, that $\mathcal{A}$ is $\mathcal{K}$-consistent and $\mathcal{K}$-complete, and that $\langle(q_1), Gq_1\rangle$ belongs to $\mathcal{A}$ ($\mathcal{K}$-complete axiom systems that contain this rule have actually been given). Then the formula $q_1 \rightarrow Gq_1$ will be provable from $\mathcal{A}'$, though it is not $\mathcal{K}$-valid. (The provability of $q_1 \rightarrow Gq_1$ from $\mathcal{A}'$ can be easily verified.) We will, however, select among the proofs from $\mathcal{A}'$ a certain group, which we will call (by lack of a better term) *sound proofs*, in such a way that if $\mathcal{A}$ is $\mathcal{K}$-consistent then all formulae of $\mathcal{L}_2$ occurring in sound proofs from $\mathcal{A}'$ will be $\mathcal{K}$-valid, whereas on the other hand, if $\mathcal{A}$ is $\mathcal{K}$-complete then the formulae of $\mathcal{L}_2$ occurring in sound proofs from $\mathcal{A}'$ will include the set of all $\mathcal{K}$-valid formulae of $\mathcal{L}_2$.

DEFINITION 18. Let $\mathcal{A}$ be an axiom system for $\mathcal{L}_1$. A proof from $\mathcal{A}'$ is *sound* if each of its members that is preceded in the proof by

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[6] For any formula φ, $L\varphi$ stands for $\varphi \wedge H\varphi \wedge G\varphi$.

an instance of N_1 or N_2 is itself an instance of N_1 or N_2, or else follows from its predecessors by MP. A formula is said to be *soundly provable from* $\mathcal{A}'$, if it occurs in a sound proof from $\mathcal{A}'$. The following facts concerning provability from $\mathcal{A}$ and sound provability from $\mathcal{A}$ and sound provability from $\mathcal{A}'$ are easily verified, and will be most useful in further developments. We state them without proof.

(1) If $\mathcal{A}$ is an axiom system for $\mathcal{L}_1$, φ is provable from $\mathcal{A}$ [soundly provable from $\mathcal{A}$] and ψ is an instance of φ, then ψ is provable from $\mathcal{A}$ [soundly provable from $\mathcal{A}$].

(2) If $\mathcal{A}$ is an axiom system for $\mathcal{L}_1$ and φ is provable from $\mathcal{A}$ then φ is soundly provable from $\mathcal{A}'$.

(3) If $\mathcal{A}$ is an axiom system for $\mathcal{L}_2$, $\varphi_1, \ldots, \varphi_n$ are soundly provable from $\mathcal{A}$, and $\varphi_1 \wedge \ldots. \wedge \varphi_n \rightarrow \varphi$ is soundly provable from $\mathcal{A}$, then φ is soundly provable from $\mathcal{A}$.

The notions of *$\mathcal{K}$-consistency in $\mathcal{L}_2$ relative to $\mathcal{A}$*, of *$\mathcal{K}$-consistency in $\mathcal{L}_2$* (of $\mathcal{A}$) and of *weak* and *strong $\mathcal{K}$-completeness in $\mathcal{L}_2$* are defined as before, except that in the definitions 'provable' should everywhere be replaced by 'soundly provable'.

Intuitively one would demand of an inference rule R which is used in an axiomatization of (all or some of) the $\mathcal{K}$-valid formulae of $\mathcal{L}_1$ that whenever ψ comes by an applidation of R from $\psi_1, \ldots, \psi_k$, and $\psi_1, \ldots, \psi_k$ are $\mathcal{K}$-valid, then so is ψ. That an axiomsystem containing R is $\mathcal{K}$-consistent in $\mathcal{L}_2$ is in itself no guarantee that R has this property. For example, the axiom system consisting only of the rule $\langle\{q_1 \rightarrow q_1\}, \sim(q_1 \rightarrow q_1)\rangle$ is $\mathcal{K}$-consistent since one can prove from it no formula whatsoever. But the rule itself clearly fails to have the mentioned property. On the other hand, $\mathcal{A}$ will indeed be $\mathcal{K}$-consistent in $\mathcal{L}_1$ if all its members have this property. We will call a rule with this property $\mathcal{K}$-valid, and an axiom system all members of which are $\mathcal{K}$-valid, *strongly $\mathcal{K}$-consistent*.

With respect to the language $\mathcal{L}_2$ matters are a little more complicated. For we just saw that this language contains valid formulae, e.g., $q_1 \leftrightarrow Nq_1$, from which we can obtain invalid formulae, as, e.g., $G(q_1 \leftrightarrow Nq_1)$ by application of a rule (in this case $\langle\{q_1\}, Gq_1\rangle$) which is perfectly acceptable in the sense discussed above when considered as a rule for $\mathcal{L}_1$. However, every inference rule for $\mathcal{L}_1$ which is $\mathcal{K}$-valid in the sense discussed above will have the property that it yields from formulae of $\mathcal{L}_2$ which are true in every model relative to every member of $\mathcal{K}$ *at every moment*, only formulae which satisfy this same condition (such formulae will be called *strongly* $\mathcal{K}$*-valid*). It is this last property which we will need in connection with $\mathcal{L}_2$. Thus we come to the following.

DEFINITION 19. (1) A formula of $\mathcal{L}_2$ is *strongly* $\mathcal{K}$*-valid* iff for every $\mathcal{T} \in \mathcal{K}$, interpretation $\langle \mathcal{M}, t_0 \rangle$ relative to $\mathcal{T}$ and $t \in T$ $[\varphi]^{\mathcal{T}}_{<\mathcal{M}, t_0>, t} = 1$. (2) An inference rule $\langle S, \varphi \rangle$ for $\mathcal{L}_1$ [for $\mathcal{L}_2$] is $\mathcal{K}$*-valid for* $\mathcal{L}_1$ [*for* $\mathcal{L}_2$] iff whenever $\psi, \psi_1, \ldots, \psi_k$ are formulae of $\mathcal{L}_1$ [of $\mathcal{L}_2$], $\psi_1, \ldots, \psi_k$ are $\mathcal{K}$-valid [strongly $\mathcal{K}$-valid] ,and ψ comes from $\psi_1, \ldots, \psi_k$ by an application of $\langle S, \varphi \rangle$ then ψ is $\mathcal{K}$-valid [strongly $\mathcal{K}$-valid]. (3) An axiomsystem $\mathcal{A}$ for $\mathcal{L}_2$ is *strongly* $\mathcal{K}$*-consistent in* $\mathcal{L}_1$ [*in* $\mathcal{L}_2$] iff all its members are $\mathcal{K}$-valid for $\mathcal{L}_1$ [for $\mathcal{L}_2$]. Note that every formula of $\mathcal{L}_1$ is strongly $\mathcal{K}$-valid if it is $\mathcal{K}$-valid; and that a rule $\langle \emptyset, \varphi \rangle$ is $\mathcal{K}$-valid iff φ is strongly $\mathcal{K}$-valid.

It is now possible to state the result which partly confirms the mentioned conjecture on completeness which occurs in Prior's article [2].

THEOREM 1. *Let* $\mathcal{K}$ *be a non-empty class of linear orderings; let* $\mathcal{A}$ *be an axiomsystem for* $\mathcal{L}_1$ *which is strongly* $\mathcal{K}$*-consistent in* $\mathcal{L}_2$ *and weakly* [*strongly*] $\mathcal{K}$*-complete in* $\mathcal{L}_1$. *Then* $\mathcal{A}'$ *is* $\mathcal{K}$*-consistent in* $\mathcal{L}_2$ *and weakly* [*strongly*] $\mathcal{K}$*-complete in* $\mathcal{L}_2$.

Theorem 1 will be proved in the next section. In fact, it will be a corollary to a number of much more general results, from which I will also derive a complete confirmation of the conjecture, made in Prior's article, that a certain axiom system presented there is complete (cf. [2], p. 113). I have stated Theorem 1 already here, as it is a paradigm of all the theorems to follow in the next two sections.

It is worth noting that if the axiom system $\mathcal{A}$ is $\mathcal{K}$-consistent and $\mathcal{K}$-complete and the only members of $\mathcal{A}$ which are not axioms are the rules MP, $\langle \{q_1\}, \mathsf{G}q_1 \rangle$ and $\langle \{q_1\}, \mathsf{H}q_1 \rangle$, then

(1) $\mathcal{A}'$ is $\mathcal{K}$-consistent and $\mathcal{K}$-complete.

(2) The rule $\langle \{q_1\}, \mathsf{L}q_1 \rangle$, and the axioms N_{L}, $\mathsf{N}_{\sim}$ and $\mathsf{N}_{\rightarrow}$ are redundant.

(1) follows from the fact that $\mathcal{A}$ is strongly $\mathcal{K}$-consistent (all its rules are $\mathcal{K}$-valid). (2) can be argued as follows. Let $\mathcal{A}'' = \mathcal{A} \cup \{\mathsf{N}_2\}$. Let P be a proof of the formula φ_0 of $\mathcal{L}_2$ from $\mathcal{A}'$. In the first place we may eliminate all applications of the rule $\langle\{\boldsymbol{q}_1\}, \mathsf{L}\boldsymbol{q}_1\rangle$ from P in view of the presence of the rules $\langle\{\boldsymbol{q}_1\}, \mathsf{G}\boldsymbol{q}_1\rangle$ and $\langle\{\boldsymbol{q}_1\}, \mathsf{H}\boldsymbol{q}_1\rangle$. Let us assume therefore that P contains no applications of $\langle\{\boldsymbol{q}_1\}, \mathsf{L}\boldsymbol{q}_1\rangle$.

Let P_1 be that part of P which precedes the first instance of N_1 in P (or all of P if no such instance occurs). First we produce a proof P' from A'' in which occurs, for every line φ of P_1, a formula of the form $\sigma_1 \to (\sigma_2 \to \ldots (\sigma_k \to \varphi) \ldots)$, where each of the σ_i is an instance of N_L, $\mathsf{N}_\sim$ or $\mathsf{N}_\to$. The construction of P' proceeds inductively down the lines of P_1 and is possible in view of the following facts:

(a) If φ is an instance of an axiom of $\mathcal{A}'$ other than N_1, N_L, $\mathsf{N}_\sim$, or $\mathsf{N}_\to$ then φ is provable from $\mathcal{A}''$.

(b) If φ is an instance of N_L, $\mathsf{N}_\sim$ or $\mathsf{N}_\to$, then $\varphi \to \varphi$ is provable from $\mathcal{A}''$.

(c) If φ comes by MP from ψ and $\psi \to \varphi$, and $\varrho_1 \to (\varrho_2 \to \ldots (\varrho_k \to \psi) \ldots)$ and $\sigma_1 \to (\sigma_2 \to \ldots (\sigma_n \to (\psi \to \varphi) \ldots)$ are provable from $\mathcal{A}''$, then $\varrho_1 \to (\varrho_2 \to \ldots \varrho_k \to (\sigma_1 \to (\sigma_2 \to \ldots (\sigma_n \to \varphi) \ldots)) \ldots)$ is provable from $\mathcal{A}''$.

(d) If φ is of the form $\mathsf{G}\psi$ and comes from ψ by an application of $\langle\{\boldsymbol{q}_1\}, \mathsf{G}\boldsymbol{q}_1\rangle$, and $\varrho_1 \to (\varrho_2 \to \ldots (\varrho_k \to \psi) \ldots)$ is provable from $\mathcal{A}''$, then $\mathsf{G}(\varrho_1 \to (\varrho_2 \to \ldots (\varrho_k \to \psi) \ldots)$ is provable from $\mathcal{A}''$, and since every formula $\mathsf{G}(\chi_1 \to \chi_2) \to (\mathsf{G}\chi_1 \to \mathsf{G}\chi_2))$ is provable from $\mathcal{A}$ (which is the case because $\mathsf{G}(\boldsymbol{q}_1 \to \boldsymbol{q}_2) \to (\mathsf{G}\boldsymbol{q}_1 \to \mathsf{G}\boldsymbol{q}_2)$ is $\mathcal{K}$-valid), $\mathsf{G}\varrho_1 \to (\mathsf{G}\varrho_2 \to \ldots (\mathsf{G}\varrho_k \to \mathsf{G}\psi) \ldots)$ is provable from $\mathcal{A}''$. But we observed under (a) that for each i $\varrho_i \to \mathsf{L}\varrho_i$ is provable from $\mathcal{A}''$. Since $\mathsf{L}\boldsymbol{q}_1 \to \mathsf{G}\boldsymbol{q}_1$ is a $\mathcal{K}$-valid formula of $\mathcal{L}_1$, $\mathsf{L}\varrho_i \to \mathsf{G}\varrho_i$ is provable, and so $\varrho_i \to \mathsf{G}\varrho_i$ is provable. It follows that $\varrho_1 \to (\varrho_2 \to \ldots (\varrho_k \to \mathsf{G}\psi) \ldots)$ is provable from $\mathcal{A}''$.

(e) If φ is of the form $\mathsf{H}\psi$ and comes by an application of the rule $\langle\{\boldsymbol{q}_1\}, \mathsf{H}\psi\rangle$ and $\varrho_1 \to (\varrho_2 \to \ldots (\varrho_k \to \psi) \ldots)$ is provable from $\mathcal{A}''$ one argues in the same way that $\mathsf{H}\varrho_1 \to (\mathsf{H}\varrho_2 \to \ldots (\mathsf{H}\varrho_k \to \varphi) \ldots)$ is provable from $\mathcal{A}''$.

For each instance χ of N_L, $\mathsf{N}_\sim$ or $\mathsf{N}_\to$ there are instances $\chi_1, \ldots,$

χ_k of N_1 such that $\chi_1 \rightarrow (\chi_2 \rightarrow \ldots (\chi_k \rightarrow \chi) \ldots)$ is provable from $\mathcal{A}''$, for the formulae $(q_1 \leftrightarrow Nq_1) \rightarrow (Lq_1 \rightarrow Nq_1)$, $(q_1 \leftrightarrow Nq_1) \wedge (\sim q_1 \leftrightarrow N\sim q_1) \rightarrow (\sim Nq_1 \leftrightarrow N\sim q_1)$, and $(q_1 \leftrightarrow Nq_1) \wedge (q_2 \leftrightarrow Nq_2) \wedge ((q_1 \rightarrow q_2) \leftrightarrow N(q_1 \rightarrow q_2)) \rightarrow (N(q_1 \rightarrow q_2) \rightarrow (Nq_1 \rightarrow Nq_2))$ are provable from $\mathcal{A}''$. Thus we can extend P' to a proof P'' in which occurs for each line ψ of P_1, a formula of the form $\sigma_1 \rightarrow (\sigma_2 \rightarrow \ldots (\sigma \rightarrow \psi) \ldots)$, where the σ_i are instances of N_1. We can then extend P'' by adding all the instances of N_1 which occur as antecedents of lines in P'', all lines which result from detaching these antecedents by applications of MP, and further all remaining lines of P. The resulting proof will be a sound proof of ψ_0 from $\mathcal{A}'' \cup \{N_1\}$.

The case just described is of some importance, since all axiom systems for $\mathcal{L}_1$ which are known (to me) have indeed no other inference rules than MP, $\langle\{q_1\}, Gq_1\rangle$ and $\langle\{q_1\}, Hq_1\rangle$.

§ 3.

It is natural to ask for a justification of the specific choice of the operators G and H (with their given semantic interpretations) as primitives for a system of tense logic. A similar question can be asked within the context of ordinary propositional calculus about the primitive truth-functional connectives of any given language. But there the answer is in all 'standard' cases rather straightforward. Many combinations of well-known truth-functional connectives (e.g., $\{\sim, \wedge\}$, $\{\sim, \vee\}$, $\{\sim, \rightarrow\}$, as well as all sets that include any of these) are *functionally complete*, in the sense that every truth-function can be expressed by a formula of a propositional language containing the connectives of that combination.

A similar justification cannot be given for the choice of the operators G and H. In fact, rather simple and natural tense operators, as, e.g., the operator 'it has been the case uninterruptedly for some time' cannot be expressed by a formula of $\mathcal{L}_1$. It is therefore desirable to consider besides $\mathcal{L}_1$ other languages which have different tense operators as primitives.

In order to give a uniform characterization of these languages, it is necessary to characterize the general notion of a 'tense-operator'. Unlike for the general concept of a truth-function there is for this notion no entirely straightforward definition. What the definition ought to be is a question which I will not discuss here exhaustively. But I will treat the problem in so far as is necessary for the analysis of the operator 'now' with which this paper is concerned.

First this: Truth-functional connectives in natural languages (e.g., the expressions 'and', 'not', 'neither ... nor' of English) are expressions with the property that the truth-value (at a given moment t) of a sentence formed by means of such an expression out of other sentences is determined completely by the truth-values (at that same moment t) of the component sentences. In so far as truth is concerned the behavior of a truth-functional—say, 2-place—connective, is completely characterized by some 2-place *truth-function*, i.e., a function which assigns to each pair of truth-values a truth-value; and thus the study of truth-functional connectives can properly be reduced to the study of truth-functions.

With tense operators of natural languages (e.g., the tenses of English, or expressions like 'it has sometimes been the case that' 'it has been the case that ... ever since ...', etc.) the situation is more complicated. Here the truth-value, at a given time t, of a sentence formed by applying a tense, or such an expression, to (an) other sentence(s) does not depend only on the truth-value(s) of that (those) sentence(s) at t, but also (in some cases even exclusively) on their truth-values at moments other than t. On the other hand one easily verifies that for the expressions which were just listed as examples *nothing more* is required to determine the truth-value of the compound sentence at t than the truth-values of the component sentence(s) at t and other moments. Thus with respect to (momentary) truth-value the behavior of such an expression is completely characterized by a function which takes both as arguments and as values 'courses of truth-values through time', i.e., functions from moments to truth-values. (If $\mathfrak{T}$ is the structure of time we will refer to functions with such arguments and values as $\mathfrak{T}$-tenses.) I maintain that this principle is an essential aspect of what should be regarded as a 'tense operator'. Therefore the study of tense operators reduces to the study of tenses, just as the study of truth-functional connectives comes down to the study of truth-functions.

The above considerations naturally lead to the following definition.

DEFINITION 20. Let $\mathcal{T}$ be a partial ordering. An *n-place $\mathcal{T}$-tense* is an n-place function from $\{0,1\}^{\mathcal{T}}$ into $\{0,1\}^{\mathcal{T}}$.[7]

The study of tenses is complicated by the fact that we do not know what the structure of time is. I said already that for this reason we want to consider classes of partial orderings, as well as single partial orderings. For given tense operators of English it is often fairly clear what the characterizing $\mathcal{T}$-tense is, independently of any exact determination of the structural properties of $\mathcal{T}$. So, for example, will the $\mathcal{T}$-tense corresponding to the expression 'it has always been the case that' be the function $F_{\mathcal{T}}$ which assigns to a function f from moments to truth-values that function g such that for any moment $t \in T$ $g(t)=1$ iff for all t' preceding t, $f(t')=1$. Thus there corresponds to this tense operator a $\mathcal{T}$-tense $F_{\mathcal{T}}$ for every possible time structure $\mathcal{T}$—i.e., a function which assigns to each possible time structure $\mathcal{T}$ a $\mathcal{T}$-tense. I take such functions to embody the most general idea of a *tense.*

DEFINITION 21. An *n-place tense* is a function which assigns to each partial ordering $\mathcal{T}$ an n-place $\mathcal{T}$-tense.

For any class of partial orderings $\mathcal{K}$, an n-place $\mathcal{K}$-tense is the restriction to $\mathcal{K}$ of an n-place tense.

It is doubtful whether every function which is a $\mathcal{T}$-tense according to Definition 20 is intuitively acceptable as the semantic characterization of a possible tense operator. Even less plausible is it that every tense can be so regarded. For example, a tense which assigns to the structure *Re* of the real numbers the *Re*-tense which corresponds to the expression 'it has always been the case that' and to the structure *Rat* of the rational numbers the *Rat*-tense corresponding to the expression 'it will some time be the case that' clearly is a monstrosity. It is therefore desirable to

[7] For any sets U, V, understand by U^V the set of all functions from V into U.

further limit the notion of a tense. I maintain that this can indeed be done in a satisfactory way. (Cf. [4].) However, as the results on 'now' which are proved in this article hold true even for the excessively general notion of a tense developed here, we will leave this problem aside.

To each n-place truth-function f corresponds a tense $\mathfrak{F}$ such that for every partial ordering $\mathcal{T}$, $t \in T$ and members $P_1, \ldots, P_n$ of $\{0,1\}^{\mathcal{T}}$, $\mathfrak{F}(P_1(t), \ldots, P_n(t)) = (\mathfrak{F}_{\mathcal{T}}(P_1, \ldots, P_n))(t)$. This fact enables us to treat truth-functional connectives—and in particular the standard connectives which occur, e.g., in $\mathfrak{L}_1$—as special tense operators.

DEFINITION 22. (1) For any countable indexed family $\mathfrak{F}$ of tenses let $\mathfrak{L}_{\mathfrak{F}}$ be the language whose connectives are those symbols C_i^n such that $i \in \mathrm{Dom}\,\mathfrak{F}$ and n is the number of places of $\mathfrak{F}(i)$.[8]
(2) The notion of an *interpretation for* $\mathfrak{L}_{\mathfrak{F}}$ *relative to* $\mathcal{T}$ is the same as that of an interpretation for $\mathfrak{L}_1$ relative to $\mathcal{T}$ as defined in Section 1 (Definition 6).
(3) The *truth-value of* a formula φ of $\mathfrak{L}_{\mathfrak{F}}$ *in* an interpretation $\langle \mathcal{M}, t_0\rangle$ *relative to* $\mathfrak{F}$ and $\mathcal{T}$ *at* a moment $t \in T$, $[\varphi]^{\mathfrak{F},\mathcal{T}}_{<\mathcal{M},t_0>,t}$, is defined by:

(i) $[q_1]^{\mathfrak{F},\mathcal{T}}_{<\mathcal{M},t_0>,t} = \mathcal{M}(q_1)(t)$;
(ii) $[C_i^n(\varphi_1 \ldots \varphi_n)]^{\mathfrak{F},\mathcal{T}}_{<\mathcal{M},t_0>,t} = \mathfrak{F}(i)(\bar{\varphi}_1, \ldots, \bar{\varphi}_n)(t)$,
where for $i = 1, \ldots, n$,
$\bar{\varphi}_i = \{t' \in \mathcal{T}: [\varphi_i]^{\mathfrak{F},\mathcal{T}}_{<\mathcal{M},t_0>,t'} = 1\}$.

(4) A formula φ of $\mathfrak{L}_{\mathfrak{F}}$ is *$\mathcal{T}$-valid, relative to* $\mathfrak{F}$, if for every interpretation $\langle \mathcal{M}, t_0\rangle$ for $\mathfrak{L}_{\mathfrak{F}}$ relative to $\mathcal{T}$, $[\varphi]^{\mathfrak{F},\mathcal{T}}_{<\mathcal{M},t_0>,t_0} = 1$. φ is *$\mathcal{K}$-valid relative to* $\mathfrak{F}$ iff φ is $\mathcal{T}$-valid, relative to $\mathfrak{F}$, for all $\mathcal{T} \in \mathcal{K}$.
(5) For any language $\mathfrak{L}_{\mathfrak{F}}$ let $\mathfrak{L}_{\mathfrak{F}}(\mathrm{N}) = \mathfrak{L}_{\mathfrak{F}} \cup \{\mathrm{N}\}$. The *interpretations for* $\mathfrak{L}_{\mathfrak{F}}(\mathrm{N})$ *relative to* $\mathcal{T}$ are the interpretations for $\mathfrak{L}_{\mathfrak{F}}$ relative to $\mathcal{T}$, and $[\varphi]^{\mathfrak{F},\mathcal{T}}_{<\mathcal{M},t_0>,t}$ is defined for formulae of $\mathfrak{L}_{\mathfrak{F}}(\mathrm{N})$ by the clauses (i) and (ii) above together with the clause

(iii) $[\mathrm{N}\varphi]^{\mathfrak{F},\mathcal{T}}_{<\mathcal{M},t_0>,t} = [\varphi]_{<\mathcal{M},t_0>,t_0}$.

$\mathfrak{F}$-validity and *$\mathcal{K}$-validity* are then defined as above.

[8] By a *countable indexed family* of objects of a certain kind I understand here a function which is defined on some set of positive natural numbers and which assigns to each member of its domain an object of the kind in question.

We will, for simplicity, assume that all the languages to be considered henceforth are truth-functionally complete. We will be even more specific and consider only indexed families $\mathfrak{J}$ of tenses the domains of which contain the numbers 1 and 4 and where $\mathfrak{J}(1)$, $\mathfrak{J}(4)$ are the tenses which correspond to the standard meaning of the connectives $\sim$, $\wedge$, respectively.

Our first result of this section states that any formula of a language $\mathcal{L}_{\mathfrak{J}}(\mathsf{N})$ which contains N is equivalent to a formula that does not contain N.

THEOREM 2. *Let $\mathfrak{J}$ be an indexed family of tenses. There is a (primitive recursive) function $R_{\mathfrak{J}}$ from the formulae of $\mathcal{L}_{\mathfrak{J}}(\mathsf{N})$ to formulae of $\mathcal{L}_{\mathfrak{J}}$, such that, for every partial ordering $\mathfrak{T}$, $\varphi \leftrightarrow R_{\mathfrak{J}}(\varphi)$ is $\mathfrak{T}$-valid relative to $\mathfrak{J}$.*

PROOF. By a *special formula* of $\mathcal{L}_{\mathfrak{J}}(\mathsf{N})$ understand a formula of the form $\bigvee_{i=1}^{k} (\varphi_i \wedge \mathsf{N}\psi_i)$, where none of the formulae φ_i, ψ_i contains N. (By $\bigvee_{i=1}^{k}\sigma_i$ understand $\sigma_1 \vee \sigma_2 \vee \ldots. \vee \sigma_k$. Similarly for $\bigvee_{\sigma\in S}\sigma$.) We first define inductively a function $R'_{\mathfrak{J}}$ which maps the formulae of $\mathcal{L}_{\mathfrak{J}}(\mathsf{N})$ into special formulae of $\mathcal{L}_{\mathfrak{J}}\mathsf{N})$:

(i) $R'_{\mathfrak{J}}(q_i) = q_i$;
(ii) suppose that $i \in \mathrm{Dom}\ \mathfrak{J}$, that $\mathfrak{J}(i)$ is n-place, that $R'_{\mathfrak{J}}(\varphi_1)$, $\ldots$, $R'_{\mathfrak{J}}(\varphi_n)$ have been defined, and that for $i = 1, \ldots, n$ $R'_{\mathfrak{J}}(\varphi_i) = \bigvee_{j=1}^{k_i} (\varphi_{ij} \wedge \mathsf{N}\psi_{ij})$.

For any $i \leq n$ let S_i be the set of all formulae $\chi_1 \wedge \ldots \wedge \chi_{k_i}$ where for $j = 1, \ldots., k_i$, χ_j is ψ_{ij} or χ_j is $\sim\psi_{ij}$. Let S be the set of all formulae of the form $(\sigma_1) \wedge \ldots \wedge (\sigma_n)$, where for $i = 1, \ldots, n$, $\sigma_i \in S_i$. For each $\sigma = \chi_1 \wedge \ldots \wedge \chi_{k_i} \in S_i$ let

$$\varphi_i(\sigma) = \begin{cases} \text{the disjunction of all those } \varphi_{ij}, \text{ for which} \\ \quad \chi_j = \psi_{ij}, \text{ provided some } \chi_j \text{ is } \psi_{ij}; \\ \sim(\varphi_{i1} \rightarrow \varphi_{i1}), \text{ if for all } j \leq k_i,\ \chi_j \text{ is } \sim\psi_{ij}. \end{cases}$$

For each $\sigma \in S$, let $\varphi_i(\sigma)$ be $\varphi_i(\sigma_i)$ where σ_i is the i^{th} conjunct of σ. We put

$$R'_{\mathfrak{J}}(\mathsf{C}_i^n(\varphi_1, \ldots, \varphi_n)) = \bigvee_{\sigma\in S}(\mathsf{C}_i^n(\varphi_1(\sigma), \ldots, \varphi_n(\sigma)) \vee \mathsf{N}\sigma).$$

(iii) Suppose that $R'_{\mathcal{J}}(\varphi) = \bigvee_{j=1}^{k} (\varphi_j \wedge N\psi_j)$. Then put
$$R'_{\mathcal{J}}(N\varphi) = \bigvee_{j=1}^{k} ((\varphi_j \rightarrow \varphi_j) \wedge N(\varphi_j \wedge \psi_j)).$$

Clearly $R'_{\mathcal{J}}$, as defined by (i)-(iii) is a function which maps every formula of $\mathcal{L}_{\mathcal{J}}(N)$ onto a special formula. It is tedious but straightforward to prove by induction that for all φ of $\mathcal{L}_{\mathcal{J}}(N)$, $\varphi \leftrightarrow R'_{\mathcal{J}}(\varphi)$ is $\mathcal{J}$-valid (for arbitrary $\mathcal{J}$).

For any formula φ of $\mathcal{L}_{\mathcal{J}}$ let $R_{\mathcal{J}}(\varphi)$ be the formula which we obtain by deleting all occurrences of N from $R'_{\mathcal{J}}(\varphi)$. Clearly $R_{\mathcal{J}}(\varphi)$ is a formula of $\mathcal{L}_{\mathcal{J}}$. It is also obvious that $R'_{\mathcal{J}}(\varphi) \leftrightarrow R_{\mathcal{J}}(\varphi)$ is $\mathcal{J}$-valid for all $\mathcal{J}$. It follows that $\varphi \leftrightarrow R_{\mathcal{J}}(\varphi)$ is $\mathcal{J}$-valid; q.e.d.

The general concepts of an *inference rule*, an *axiom*, an *axiom system* for a language $\mathcal{L}_{\mathcal{J}}$, as well as the notion of *provability*, have already been given on pp. 0.0, 0.0. We now define for an arbitrary axiom system $\mathcal{A}$ for $\mathcal{L}_{\mathcal{J}}$ an extension $\mathcal{A}_j$ which will stand to $\mathcal{A}$ in the same relation as, for any axiom system $\mathcal{B}$ for $\mathcal{L}_1$, the system $\mathcal{B}'$ for $\mathcal{L}_2$ stands to $\mathcal{B}$.

DEFINITION 23. Let $\mathcal{J}$ be an indexed family of tenses.
(1) Let $\mathcal{N}(\mathcal{J})$ be the set containing
(a) all pairs

$$\langle \emptyset, C_i^n(q_1 \ldots q_{j-1} ((Nq_j \wedge q_{j+1}) \vee q_{j+2})\, q_{j+3} \ldots q_{n+2}) \leftrightarrow [(Nq_j \wedge C_i^n(q_1 \ldots q_{j-1} (q_{j+1} \vee q_{j+2})\, q_{j+3} \ldots q_{n+2})) \vee N \sim q_j \wedge C_i^n(q_1 \ldots q_{j-1}\, q_{j+2}\, q_{j+3} \ldots q_{n+2})] \rangle,$$

where $i \in \mathrm{Dom}\, \mathcal{J}$, n equals the number of places of $\mathcal{J}(i)$ and $1 \leq j \leq n$;
(b) all pairs $\langle \{q_1 \leftrightarrow q_2\}, \varphi \leftrightarrow \varphi' \rangle$ where φ is a formula of $\mathcal{L}_{\mathcal{J}}$ and φ' results from replacing an occurrence of q_1 in φ by q_2.
(2) For any axiomsystem $\mathcal{A}$ for $\mathcal{L}_{\mathcal{J}}$ let $\mathcal{A}^* = \mathcal{A} \cup \mathcal{N}(\mathcal{J}) \cup \{\mathrm{MP}, N_{\sim}, N_1, N_2\}$. Again a *sound proof* from $\mathcal{A}^*$ is a proof in which every line following an instance of N_1 is itself an instance of N_1 or else comes from previous lines by MP. The concepts of *$\mathcal{K}$-consistency, strong $\mathcal{K}$-consistency, weak $\mathcal{K}$-completeness* and *strong $\mathcal{K}$-completeness* in $\mathcal{L}_{\mathcal{J}}$, or $\mathcal{L}_{\mathcal{J}}(N)$, of $\mathcal{A}$ or $\mathcal{A}^*$ (where $\mathcal{A}$ is an axiom system for $\mathcal{L}_{\mathcal{J}}$) are straightforward generalizations of the notions defined earlier for $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{A}$ and $\mathcal{A}'$; I will not spell them out.

The next theorem is similar to Theorem 1, but has a much wider range of application.

THEOREM 3. *Let $\mathcal{K}$ be a non-empty class of partial orderings; let $\mathfrak{I}$ be an indexed set of tenses; let $\mathcal{A}$ be an axiom system for $\mathfrak{L}_{\mathfrak{I}}$ which is strongly $\mathcal{K}$-consistent in $\mathfrak{L}_{\mathfrak{I}}(\mathsf{N})$ and weakly [strongly] $\mathcal{K}$-complete in $\mathfrak{L}_{\mathfrak{I}}$, Then $\mathcal{A}^*$ is $\mathcal{K}$-consistent in $\mathfrak{L}_{\mathfrak{I}}(\mathsf{N})$ and weakly [strongly] $\mathcal{K}$-complete in $\mathfrak{L}_{\mathfrak{I}}(\mathsf{N})$.*

PROOF. One easily verifies that if $\mathcal{A}$ is weakly $\mathcal{K}$-complete in $\mathfrak{L}_{\mathfrak{I}}$ then $\varphi \leftrightarrow R_{\mathfrak{I}}(\varphi)$ is soundly provable from $\mathcal{A}^*$. For $\varphi \leftrightarrow R'_{\mathfrak{I}}(\varphi)$ (where $R'_{\mathfrak{I}}$ defined as in the proof of Theorem 2) is provable from $\mathcal{A}^* - \{\mathsf{N}_1\}$, and $R'(\varphi) \leftrightarrow R(\varphi)$ is soundly provable from $\mathcal{A}^*$. Suppose first that $\mathcal{A}$ is weakly $\mathcal{K}$-complete in $\mathfrak{L}_{\mathfrak{I}}$. Let φ be a $\mathcal{K}$-valid formula of $\mathfrak{L}_{\mathfrak{I}}(\mathsf{N})$. Then, by Theorem 2, $R_{\mathfrak{I}}(\varphi)$ is $\mathcal{K}$-valid. Since $R_{\mathfrak{I}}(\varphi)$ is a formula of $\mathfrak{L}_{\mathfrak{I}}$ and $\mathcal{A}$ is $\mathcal{K}$-complete in $\mathfrak{L}_{\mathfrak{I}}$, $R_{\mathfrak{I}}(\varphi)$ is provable from $\mathcal{A}$. Thus by the above observation φ is soundly provable from $\mathcal{A}^*$; and thus $\mathcal{A}^*$ is weakly $\mathcal{K}$-complete in $\mathfrak{L}_{\mathfrak{I}}(\mathsf{N})$.

Now suppose that $\mathcal{A}$ is strongly $\mathcal{K}$-complete in $\mathfrak{L}_{\mathfrak{I}}$. Let Δ be a set of formulae of $\mathfrak{L}_{\mathfrak{I}}(\mathsf{N})$ which is consistent relative to $\mathcal{A}^*$. Let $R_{\mathfrak{I}}(\Delta)$ be the set of all $R_{\mathfrak{I}}(\varphi)$ where $\varphi \in \Delta$. Since $\varphi \leftrightarrow R_{\mathfrak{I}}(\varphi)$ is soundly provable from $\mathcal{A}^*$ for all $\varphi \in \Delta$, $R_{\mathfrak{I}}(\Delta)$ is consistent relative to $\mathcal{A}$. Thus there is a $\mathcal{T} \in \mathcal{K}$ and an interpretation $\langle \mathcal{M}, t_0 \rangle$ relative to $\mathcal{T}$ such that for all $\psi \in R_{\mathfrak{I}}(\Delta)$, $[\psi]^{\mathfrak{I}, \mathcal{T}} = 1$. So, by Theorem 2, $[\varphi]^{\mathfrak{I}, \mathcal{T}}_{<\mathcal{M}, t_0>, t_0} = 1$ for all $\varphi \in \Delta$.

To show that $\mathcal{A}^*$ is $\mathcal{K}$-consistent in $\mathfrak{L}_{\mathfrak{I}}(\mathsf{N})$ we first show by induction on the length of proofs that every formula which is provable from $\mathcal{A}^* - \{\mathsf{N}_1\}$ is strongly $\mathcal{K}$-valid. This is true since all inference rules in $\mathcal{A}^* - \{\mathsf{N}_1\}$ (including the axioms!) are strongly $\mathcal{K}$-valid. Thus a sound proof from $\mathcal{A}^*$ will always consist of a number of lines which are (strongly) $\mathcal{K}$-valid, followed by a number of instances of N_1 and applications of MP. It is clear that N_1 is $\mathcal{K}$-valid and that MP preserves $\mathcal{K}$-validity. Thus all soundly provable formulae are $\mathcal{K}$-valid.

THEOREM 4. *Let $\mathcal{K}$, $\mathfrak{I}$, $\mathcal{A}$ be as in Theorem 3. Suppose there is a formula λ of $\mathfrak{L}_{\mathfrak{I}}$ such that of every $\mathcal{T} \in \mathcal{K}$, interpretation $\langle \mathcal{M}, t_0 \rangle$ relative to $\mathcal{T}$ and $t \in \mathcal{T}$,*

$$[\lambda]^{\mathfrak{I},\mathcal{T}}_{<\mathfrak{m},t_0>,\,t} = 1 \text{ iff for all } t' \in \mathcal{T},\ [q_1]^{\mathfrak{I},\mathcal{T}}_{<\mathfrak{m},j_0>,\,t'} = 1.$$

(We will write '$\mathsf{L}\varphi$*' for* $[\lambda]\ \varphi/q_2$.) *Let* $\mathcal{A}' = \mathcal{A} \cup \{\text{MP}, \mathsf{N}_1, \mathsf{N}_2, \mathsf{N}_{\mathsf{L}}, \mathsf{N}_{\sim}, \mathsf{N}_{\rightarrow}, \langle\{q_1\}, \mathsf{L}q_1\rangle\}$. *If* $\mathcal{A}$ *is weakly [strongly]* $\mathcal{K}$*-complete in* $\mathcal{L}_{\mathfrak{I}}$*, then* $\mathcal{A}'$ *is weakly [strongly]* $\mathcal{K}$*-complete in* $\mathcal{L}_{\mathfrak{I}}(\mathsf{N})$.

PROOF. Every axiom in $\mathcal{N}(\mathcal{F})$ is provable from $\mathcal{A}' - \{\mathsf{N}_1\}$. For

$$\mathsf{L}(q_{j+1} \rightarrow \mathsf{L}q_{j+1}) \rightarrow [\mathsf{C}^n_i(q_1 \dots q_{j-1}\,((q_j \wedge q_{j+1}) \vee q_{j+2})\, q_{j+3} \dots q_{n+2}) \leftrightarrow$$
$$[(q_{j+1} \wedge \mathsf{C}^n_i(q_1 \dots q_{j-1}\,(q_j \vee q_{j+2})\, q_{j+3} \dots q_{n+2})) \vee$$
$$(\sim q_{j+1} \wedge \mathsf{C}^n_i(q_1 \dots q_{j-1}\, q_{j+2}\, q_{j+3} \dots q_{n+2}))]]$$

is a $\mathcal{K}$-valid formula of $\mathcal{L}_{\mathfrak{I}}$, and thus all its instances are provable from $\mathcal{A}$. Since moreover $\mathsf{L}((\mathsf{N}q_{j+1} \rightarrow \mathsf{LN}q_{j+1})$ and $\sim\mathsf{N}q_{j+1} \leftrightarrow \mathsf{N}\sim q_{j+1}$ are provable from $\mathcal{A}'' - \{\mathsf{N}_1\}$, it follows that

$$\mathsf{C}^n_i(q_1 \dots q_{j-1}\,((q_j \wedge \mathsf{N}q_{j+1}) \vee q_{j+2})\, q_{j+3} \dots q_{n+2}) \leftrightarrow$$
$$[(\mathsf{N}q_{j+1} \wedge \mathsf{C}^n_i(q_1 \dots q_{j-1}\,(q_j \vee q_{j+2})\, q_{j+3} \dots q_{n+2})) \vee$$
$$(\mathsf{N}\sim q_{j+1} \wedge \mathsf{C}^n_i(q_1 \dots q_{j-1}\, q_{j+2}\, q_{j+3} \dots q_{n+2}))]$$

is provable from $\mathcal{A}' - \{\mathsf{N}_1\}$.

Further we observe that for all formulae φ, φ' of $\mathcal{L}_{\mathfrak{I}}(\mathsf{N})$, where φ' comes from φ by replacement of an occurrence of q_1 by q_2, $\mathsf{L}(q_1 \leftrightarrow q_2) \rightarrow \mathsf{L}(\varphi \leftrightarrow \varphi')$ is provable from $\mathcal{A}' - \{\mathsf{N}_1\}$. The proof is by induction on the number of occurrences of N in φ. If φ does not contain N, then $\mathsf{L}(q_1 \leftrightarrow q_2) \rightarrow \mathsf{L}(\varphi \leftrightarrow \varphi')$ is a $\mathcal{K}$-valid formula of $\mathcal{L}^{\mathfrak{I}}$ and thus provable from $\mathcal{A}$. Now assume the assertion has been demonstrated for formulae with at most k occurrences of N and suppose that φ contains $k+1$ such occurrences. Let Oc be a particular occurrence of q_1 in φ and let φ' be the formula which results if we replace Oc by q_2. Let n be the maximum of the indices of propositional constants occurring in φ or φ'. If φ has a subformula of the form $\mathsf{N}\chi$ which does not contain Oc, let ψ be the formula which we obtain by replacing that subformula by q_{n+1}, and ψ' the result of replacing Oc in ψ by q_2. By induction hypothesis $\mathsf{L}(q_1 \leftrightarrow q_2) \rightarrow \mathsf{L}(\psi \leftrightarrow \psi')$ is provable from $\mathcal{A}' - \{\mathsf{N}_1\}$. Since $\mathsf{L}(q_1 \leftrightarrow q_2) \rightarrow \mathsf{L}(\varphi \leftrightarrow \varphi')$ is an instance of this formula it too must be provable from $\mathcal{A}' - \{\mathsf{N}_1\}$. If all occurrences of N in φ are in front of subformulae containing Oc, then φ must have a subformula $\mathsf{N}\varrho$ where

ϱ contains Oc but no occurrence of N. Let ϱ' be the result of replacing Oc in ϱ by q_2. Then $\mathsf{L}(q_1 \leftrightarrow q_2) \rightarrow \mathsf{L}(\varrho \leftrightarrow \varrho')$ is provable from $\mathcal{A}$. Further, with the help of $\mathsf{N}_{\sim}$, $\mathsf{N}_{\rightarrow}$ and $\langle \emptyset, \mathsf{L}q_1 \rightarrow \mathsf{N}q_1 \rangle$ we can prove $\mathsf{L}(\varrho \leftrightarrow \varrho') \rightarrow \mathsf{L}(\mathsf{N}\varrho \leftrightarrow \mathsf{N}\varrho')$. Thus $\mathsf{L}(q_1 \leftrightarrow q_2) \rightarrow \mathsf{L}(\mathsf{N}\varrho \leftrightarrow \mathsf{N}\varrho')$ is provable from $\mathcal{A}' - \{\mathsf{N}_1\}$. Let ψ be the result of replacing $\mathsf{N}\varrho$ in φ by q_{n+1} and let ψ' be the same formula with q_{n+2} instead of q_{n+1}. Then $\mathsf{L}(q_{n+1} \leftrightarrow q_{n+2}) \rightarrow \mathsf{L}(\psi \leftrightarrow \psi')$ is by induction hypothesis provable from $\mathcal{A}' - \{\mathsf{N}_1\}$, and so is therefore its instance $\mathsf{L}(\mathsf{N}\varrho \leftrightarrow \mathsf{N}\varrho') \rightarrow \mathsf{L}(\varphi \leftrightarrow \varphi')$. It follows that $\mathsf{L}(q_1 \leftrightarrow q_2) \rightarrow \mathsf{L}(\varphi \leftrightarrow \varphi')$ is provable from $\mathcal{A}' - \{\mathsf{N}_1\}$.

It is now obvious that we can produce for any proof P from $\mathcal{A}^* - \{\mathsf{N}_1\}$ a proof P' in which, for every line φ of P, occur both φ and $\mathsf{L}\varphi$. Every step in P can either be automatically imitated in P' or else, in view of the preceding remarks, in the case of an application of a rule in $\mathcal{N}(\mathcal{J})$ can be replaced by a proof from $\mathcal{A}' - \{\mathsf{N}_1\}$ of the appropriate theorem and, if necessary, some applications of MP. A sound proof P from $\mathcal{A}^*$ can then be converted into a sound proof from $\mathcal{A}'$ because we can first convert the part of P preceding the first instance of N_1 into a proof P' from $\mathcal{A}' - \{\mathsf{N}_1\}$ and then simply add to P' the remaining lines of P. Q.e.d.

COROLLARY. Theorem 1.

PROOF. If $\mathcal{K}$ consists of linear orderings then the formula $\mathsf{H}q_1 \vee q_1 \vee \mathsf{G}q_1$ satisfies the condition imposed on the formula λ in Theorem 4. So Theorem 4 applies to $\mathcal{L}_1$.

Prior conjectures in his article on 'now' that a certain axiom system—consisting of the axioms A1.1, A1.2, A2.1, A2.2 (p. 106); the rule MP; the rule RL (p. 111); the axioms L1 –L5 (p. 111); and the axioms J1 –J6 (p. 113)—is complete. The notion of validity which Prior has in mind is that of being true (at the present) in every interpretation relative to any structure $\langle T, R \rangle$ where R is a binary relation on, but not necessarily a partial ordering of, T. (Cf. [2], p. 113.)

Indeed, for the class $\mathcal{K}_1$ of all *partial orderings* his axiom system is *not* complete, for the formula $\mathsf{G}q_1 \rightarrow \mathsf{G}\mathsf{G}q_1$, which is $\mathcal{K}_1$-valid, cannot be proved from that system. However, if we add this formula as well as its counterpart for the past, $Hq_1 \to HHq_1$, to his system we obtain an axiom system which is indeed $\mathcal{K}_1$-complete in the language which Prior discusses.

To be precise, let *Alw* be the 1-place tense which assigns to each partial ordering $\mathcal{T}$ the function $Alw(\mathcal{T})$ defined in the following way: (i) if f in $\{0, 1\}^T$ takes the value 0 for some $t \in T$, then $Alw(\mathcal{T})(f)$ is the function in $\{0, 1\}^T$ with constant value 0; (ii) if f is the function in $\{0, 1\}^T$ with constant value 1, then $Alw(\mathcal{T})(f) = f$. Let $\mathfrak{J}_0$ be an indexed set of tenses with domain consisting of the numbers 1, ..., 5 which 'assigns' to the connectives $\sim$, $\wedge$, G, H their intended meanings and such that $\mathfrak{J}_0(5) = Alw$. Let us write 'L' for C^1_5. Let $\mathcal{A}_0$ be the axiom system for $\mathcal{L}_{\mathfrak{J}_0}$ consisting of the rules MP and $\langle\{q_1\}, Lq_1\rangle$, and the axioms $G(q_1 \to q_2) \to (Gq_1 \to Gq_2)$, $H(q_1 \to q_2) \to (Hq_1 \to Hq_2)$, $PGq_1 \to q_1$, $FHq_1 \to q_1$, $Lq_1 \to q_1$, $Lq_1 \to Gq_1$, $Lq_1 \to Hq_1$, $L(q_1 \to q_2) \to (Lq_1 \to Lq_2)$, $\sim Lq_1 \to L \sim Lq_1$, $Lq_1 \to LLq_1$, together with a complete set of axioms for ordinary propositional logic. Let $\mathcal{A}_1$ be $\mathcal{A}_0$ together with the axioms $Gq_1 \to GGq_1$ and $Hq_1 \to HHq_1$. Let $\mathcal{A}_0^+$ $[\mathcal{A}_1^+]$ be the axiom system for $\mathcal{L}_{\mathfrak{J}_0}(N)$ consisting of $\mathcal{A}_0$ $[\mathcal{A}_1]$ together with the axioms $q_1 \to Nq_1$, $Nq_1 \to q_1$, $L(Lq_1 \to Nq_1)$, $L(Nq_1 \to Lq_1)$, $L(N \sim q_1 \to \sim Nq_1)$, $L(\sim Nq_1 \to N \sim q_1)$ and $L(N(q_1 \to q_2) \to (Nq_1 \to Nq_2))$. One easily verifies that $\mathcal{A}_1$ is strongly $\mathcal{K}$-consistent in $\mathcal{L}_{\mathfrak{J}_0}(N)$, that Lq_1 satisfies the condition on λ of Theorem 4, and that for $i = 0,1$ $\mathcal{A}_i'$ (as defined on p. 243) is included in $\mathcal{A}_i^+$. Moreover, it is known that $\mathcal{A}_1$ is strongly $\mathcal{K}_1$-complete in $\mathcal{L}_{\mathfrak{J}_0}$. (Cf. [5].) Thus it follows by Theorem 4 that $\mathcal{A}_1^+$ is strongly $\mathcal{K}_1$-complete in $\mathcal{L}_{\mathfrak{J}_0}(N)$.

As a matter of fact, we can just as easily obtain a complete confirmation of Prior's conjecture. It suffices to observe that the restriction to partial orderings—observed throughout this paper for reasons mentioned in Section 1—plays no role whatever in the proofs. Thus Theorem 4 is equally true if we weaken the hypothesis that $\mathcal{K}$ is a class of partial orderings to the supposition that $\mathcal{K}$ is an arbitrary class of structures $\langle T, < \rangle$ where T is a non-empty set and $<$ an (arbitrary) binary relation on T. Let $\mathcal{K}_0$ be the class of all such structures. It is known that $\mathcal{A}_0$ is strongly $\mathcal{K}_0$-complete in $\mathcal{L}_{\mathfrak{J}_0}$. (Cf. [5]). Thus $\mathcal{A}_0^+$ is strongly $\mathcal{K}_0$-complete in $\mathcal{L}_{\mathfrak{J}_0}(N)$.

§ 4.

We now turn to languages for first order predicate tense logic. For the remainder of this paper we assume that the class S mentioned in Section 1 contains besides the symbols already listed there a denumerable set of symbols $v_0, v_1, v_2, \ldots$ (called 'individual variables'); for each $n \leqq 1$ an uncountable number of symbols Q^n_α (called 'n-place predicate symbols'); a proper class of symbols c_α (called 'individual constants'); the symbols $\forall$ and $\exists$ (called the 'universal quantifier' and the 'existential quantifier', respectively); and the symbol $=$ (called the 'equality sign'). For any countable (possibly empty) set Q of predicate symbols and individual constants, and any indexed family of tenses $\mathfrak{J}$, $\mathfrak{L}_{\mathfrak{J}, Q}$ is the set consisting of the symbols of $\mathfrak{L}_{\mathfrak{J}}$ together with those of Q, the individual variables, the quantifiers, and $=$. We put $\mathfrak{L}_{\mathfrak{J}, Q}(\mathsf{N}) = \mathfrak{L}_{\mathfrak{J}, Q} \cup \{\mathsf{N}\}$. By $\mathfrak{L}_{1, Q}$ we understand the set $\mathfrak{L}_{\mathfrak{J}_0, Q}$ where $\mathfrak{J}_0$ is the family of tenses such that $\mathfrak{L}_{\mathfrak{J}_0}$ is the language $\mathfrak{L}_1$ (i.e., where $\mathfrak{J}_0$ gives the appropriate interpretations for the connectives and tense operators of $\mathfrak{L}$). The *formulae* of $\mathfrak{L}_{\mathfrak{J}, Q}$ are defined by:

(i) q_i is a *formula*;
(ii) if $Q^n_\alpha \in Q$ and $\beta_1, \ldots, \beta_n$ are either variables or individual constants in Q, then $Q^n_\alpha(\beta_1, \ldots \beta_n)$ is a *formula*;
(iii) if α and β are variables or individual constants in Q, then $\alpha = \beta$ is a *formula*;
(iv) if $i \in \mathrm{Dom}\ \mathfrak{J}$, $\mathfrak{J}(i)$ is n-placed and $\varphi_1, \ldots, \varphi_n$ are *formulae*, then $\mathsf{C}^n_i(\varphi_1 \ldots \varphi_n)$ is a *formula*;
(v) if φ is a *formula*, then $(\forall v_i)\varphi$ and $(\exists v_i)\varphi$ are *formulae*.

The formulae of $\mathfrak{L}_{\mathfrak{J}, Q}(\mathsf{N})$ are defined by clauses (i)-(v) above together with the clause

(vi) if φ is a *formula*, then $\mathsf{N}\varphi$ is a *formula*.

An occurrence of a variable α in a formula φ is *bound* if it is within a subformula of φ which is of the form $(\forall\alpha)\psi$ or $(\exists\alpha)\psi$. Other occurrences of variables are called *free*. A formula without free occurrences is called a *sentence*. The *universal closure* of a formula φ is the sentence $(\forall a_1) \ldots (\forall a_n)\ \varphi$ where $a_1, \ldots, a_n$ are all the variables which have free occurrences in φ, listed according to the magnitudes of their indices.

Notice that the formulae of $\mathfrak{L}_{\mathcal{I}}$ and $\mathfrak{L}_{\mathcal{I},\emptyset}$ coincide. More important is it to observe that if Q contains infinitely many individul constants then we can give a recursive characterization of all sentences of $\mathfrak{L}_{\mathcal{I},Q}$ (as well as of those of $\mathfrak{L}_{\mathcal{I},Q}(\mathsf{N})$) without reference to formulae which are not sentences as follows:

(i′) q_i is a *sentence of* $\mathfrak{L}_{\mathcal{I},Q}$;
(ii′) if $Q^n_a \in Q$ and $\beta_1, \ldots, \beta_n$ are individual constants in Q then $Q^n_a(\beta_1 \ldots \beta_n)$ is a *sentence of* $\mathfrak{L}_{\mathcal{I},Q}$;
(iii′) if α and β are individual constants in Q then $\alpha = \beta$ is a *sentence of* $\mathfrak{L}_{\mathcal{I},Q}$;
(iv′) if $i \in \text{Dom } \mathcal{I}$, $\mathcal{I}(i)$ is n-placed and $\varphi_1, \ldots, \varphi_n$ are *sentences of* $\mathfrak{L}_{\mathcal{I},Q}$ then $\mathsf{C}^n_i(\varphi_1 \ldots \varphi_n)$ is a *sentence of* $\mathfrak{L}_{\mathcal{I},Q}$;
(v′) if φ is a *sentence of* $\mathfrak{L}_{\mathcal{I},Q}$ and α an individual constant in Q, then $(\forall v_i)\ [\varphi]\ v_i/\alpha$ and $(\exists v_i)\ [\varphi]\ v_i/\alpha$ are *sentences of* $\mathfrak{L}_{\mathcal{I},Q}$.[9]

An *interpretation for* a language $\mathfrak{L}_{\mathcal{I},Q}(\mathsf{N})$ *relative to* $\mathcal{T}$ is a triple $\langle U, R, t\rangle$, where:

(i) U is a non-empty set;
(ii) R is a function with domain consisting of Q together with the symbols $q_1, q_2, \ldots$;
(iii) $R(q_i)$ is a function from T into $\{0, 1\}$;
(iv) if Q^n_a is an n-place predicate constant in Q, then $R(Q^n_a)$ is a function from T into $\mathfrak{P}(U^n)$;[10]
(v) if c_a is an individual constant in Q, then $R(c_a)$ is a function from T into U;
(vi) $t \in T$.

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[9] By $[\varphi]\beta/\alpha$ I understand the result of proper substitution of β for α in φ.
[10] For any set U, U^n is the nth Cartesian power of U, and $\mathfrak{P}(U)$ is the power set of U.

A *referentially complete interpretation* for $\mathfrak{L}_{\mathfrak{I},\mathcal{Q}}(\mathrm{N})$ *relative to* $\mathcal{J}$ is a quadruple $\langle U, R, H, t_0\rangle$ where $\langle U, R, t_0\rangle$ is an interpretation for $\mathfrak{L}_{\mathfrak{I},\mathcal{Q}}(\mathrm{N})$ relative to $\mathcal{J}$, and H is a $1-1$ function from a set of individual constants which are not in $\mathcal{Q}$ to constant functions from T into U.[11]

Let $\mathfrak{M} = \langle U, R, H, t_0\rangle$ be a referentially complete interpretation for $\mathfrak{L}_{\mathfrak{I},\mathcal{Q}}(\mathrm{N})$ relative to $\mathcal{J}$. For any sentence φ of $\mathfrak{L}_{\mathfrak{I},\mathcal{Q}\cup \mathrm{Dom}\, H}(\mathrm{N})$ the *truth-value relative to* $\mathfrak{I}$, $\mathcal{J}$ *in* $\mathfrak{M}$ *at t*, $[\varphi]^{\mathfrak{I},\mathcal{J}}_{\mathfrak{M},t}$ is defined by the following clauses:

(i) $[q_i]^{\mathfrak{I},\mathcal{J}}_{\mathfrak{M},t} = R(q_i)(t)$;
(ii) $[Q^n_a(c_{a_1} \ldots c_{a_n})]^{\mathfrak{I},\mathcal{J}}_{\mathfrak{M},t} = 1$ iff $\langle (R\cup H)(c_{a_1})(t), \ldots, (R\cup H)(c_{a_n})(t)\rangle \in R(Q^n_a)(t)$;
(iii) $[c_a = c_\beta]^{\mathfrak{I},\mathcal{J}}_{\mathfrak{M},t} = 1$ iff $(R\cup H)(c_a)(t) = (R\cup H)(c_\beta)(t)$;
(iv) $[\mathrm{C}^n_i(\varphi_1 \ldots \varphi_n]^{\mathfrak{I},\mathcal{J}}_{\mathfrak{M},t} = (\mathfrak{I}(i)(\bar{\varphi}_1, \ldots, \bar{\varphi}_n))(t)$, where for $j = 1, \ldots, n$ $\bar{\varphi}_j = \{t' \in T : [\varphi_j]^{\mathfrak{I},\mathcal{J}}_{\mathfrak{M},t'} = 1\}$;
(v) $[(\exists v_i)[\varphi]\, v_i/a]^{\mathfrak{I},\mathcal{J}}_{\mathfrak{M},t} = 1$ iff for some $\beta \in \mathrm{Dom}\, H$, $[[\varphi]\,\beta/a]^{\mathfrak{I},\mathcal{J}}_{\mathfrak{M},t} = 1$; $[(\forall v_i)[\varphi]\, v_i/a]^{\mathfrak{I},\mathcal{J}}_{\mathfrak{M},t} = 1$ iff for all $\beta \in \mathrm{Dom}\, H$, $[[\varphi]\,\beta/a]^{\mathfrak{I},\mathcal{J}}_{\mathfrak{M},t} = 1$;
(vi) $[\mathrm{N}\varphi]^{\mathfrak{I},\mathcal{J}}_{\mathfrak{M},t} = [\varphi]^{\mathfrak{I},\mathcal{J}}_{\mathfrak{M},t_0}$.

Now let $\mathfrak{M} = \langle U, R, t_0\rangle$ be an (arbitrary) interpretation for $\mathfrak{L}_{\mathfrak{I},\mathcal{Q}}(\mathrm{N})$, relative to $\mathcal{J}$. Then for any sentence φ of $\mathfrak{L}_{\mathfrak{I},\mathcal{Q}}(\mathrm{N})$ and $t \in T$ the *truth-value of* φ *at t in* $\mathfrak{M}$ *relative to* $\mathcal{J}$, $[\varphi]^{\mathcal{J}}_{\mathfrak{M},t}$, is the truth value of φ at t in any referentially complete interpretation $\langle U, R, H, t_0\rangle$, relative to $\mathcal{J}$.

I will not give here any philosophical justification for the treatment of individuals and quantification for tense logic which is implicit in the preceding formal definitions. I only want to say that in my opinion the system given here is certainly the most natural of those I have seen; and that it is an adequate frame for most philosophical investigations in this area. The decision to limit attention to predicate symbols and individual constants—

[11] A *constant* function from T into U is a function f from T into U for which there is a $u \in U$ such that $f(t) = u$ for all $t \in T$.

instead of considering also function symbols of arbitrarily many places—was dictated by considerations of convenience only. Theorem 5 below holds also for languages which do contain function symbols.

A formula φ of $\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$ is *valid relative to* $\mathcal{I}$ iff for every interpretation $\mathfrak{M}=\langle U, R, t_0\rangle$ for $\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$, $[\psi]^{\mathcal{I},\mathcal{I}}_{\mathfrak{M},t_0}=1$, where ψ is the universal closure of φ. $\mathcal{K}$*-validity* is defined as before. For language $\mathcal{L}_{\mathcal{I},Q}$ the definitions of an *interpretation* and the *truth-value* of a sentence *at* a moment *in* an interpretation, and *validity* of a formula are the same as those given above for $\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$, the only difference being that now clause (vi) of the truth definition above is superfluous.

An *inference rule* for a language $\mathcal{L}_{\mathcal{I},Q}$ [of $\mathcal{L}_{\mathcal{K},Q}(\mathsf{N})$] is again a pair $\langle\Sigma, \varphi\rangle$ where Σ is a finite set of formulae of $\mathcal{L}_{\mathcal{I},Q}$ [of $\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$] and φ a formula of $\mathcal{L}_{\mathcal{I},Q}$ [of $\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$]. The notions of an *axiom* and of an *axiom system* are defined as before. We say that a formula ψ of $\mathcal{L}_{\mathcal{I},Q}$ [of $\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$] *comes from* the set of formulae Σ of $\mathcal{L}_{\mathcal{I},Q}$ [of $\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$] *by an application of* the inference rule $\langle\Sigma, \varphi\rangle$ for $\mathcal{L}_{\mathcal{I},Q}$ (for $\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$] if there are variables $x_1, \ldots, x_k, x'_1, \ldots, x'_k$, predicate letters $Q_1^{i(1)}, \ldots, Q_n^{i(n)}$, and formulae $\chi_1, \ldots, \chi_n$, of $\mathcal{L}_{\mathcal{I},Q}$ [of $\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$] such that ψ comes by proper substitution in φ of x'_1 for $x_1, \ldots,$ of x'_k for x_k, of χ_1 for $Q_1^{i(1)}, \ldots,$ and of χ_n for $Q_n^{i(n)}$, and for each $\varphi' \in \Sigma$ there is a ψ' in Σ which comes by proper substitution in φ' of x'_1 for $x_1, \ldots,$ of x'_k for x_k, of χ_1 for $Q_1^{i(1)}, \ldots,$ and of χ_n for $Q_n^{i(n)}$.[12] A *proof in* $\mathcal{L}_{\mathcal{I},Q}$ [$\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$] from an axiom system A for $\mathcal{L}_{\mathcal{I},Q}$ [$\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$] is a finite sequence of formulae of $\mathcal{L}_{\mathcal{I},Q}$ [of $\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$] in which each formula either comes by one of the inference rules of $\mathcal{A}$ from the preceding formulae, or else is an alphabetic variant of a preceding formula.[12] The notion of *strong* $\mathcal{K}$*-validity* of a rule for $\mathcal{L}_{\mathcal{I},Q}$ [$\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$] and of $\mathcal{K}$*-consistency in* $\mathcal{L}_{\mathcal{I},Q}$ [$\mathcal{L}_{\mathcal{I},Q}(\mathsf{N})$], *strong* $\mathcal{K}$*-consistency in* $\mathcal{L}_{\mathcal{I},Q}$ [$\mathcal{I}_{\mathcal{I},Q}(\mathsf{N})$], and

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[12] The notions of proper substitution of a variable for a variable, of proper substitution of a formula for a predicate letter, and of an alphabetic variant, referred to here, are the usual ones. I do not give their rather involved definitions which can be found for example in Kalish and Montague [3], pp. 148, 157—159, 155—156.

weak and *strong* $\mathcal{K}$*-completeness* of an axiom system $\mathcal{A}$ for $\mathcal{L}_{\mathfrak{I},Q}$ [$\mathcal{L}_{\mathfrak{I},Q}(\mathsf{N})$] are defined as before. For any axiom system $\mathcal{A}$ for $\mathcal{L}_{\mathfrak{I},Q}$ let

$$\mathcal{A}' = \mathcal{A} \cup \{\text{MP}, \langle\{\varphi\}, (\forall v_0)\,\varphi\rangle, \mathsf{N}_1, \mathsf{N}_2\}.$$

A *sound proof* in $\mathcal{L}_{\mathfrak{I},Q}(\mathsf{N})$ from $\mathcal{A}'$ is a proof in $\mathcal{L}_{\mathfrak{I},Q}$ from $\mathcal{A}'$ in which every line following an instance of N_1 is itself such an instance, or else comes from previous lines by an application of MP or $\langle\{\varphi\}, (\forall v_0)\,\varphi\rangle$.

Our result for languages for first order predicate logic will be somewhat less general than those for languages of propositional tense logic. On the other hand, they are in my opinion more interesting, since the fact expressed by Theorem 2, that every formula containing N is equivalent to one which does not contain N, fails for languages of predicate tense logic.

THEOREM 5. *Let* $\mathcal{K}$ *be a non-empty class of partial orderings,* $\mathfrak{I}$ *an indexed set of tenses,* Q *a countable set of individual constants and predicate symbols. Let* λ *be a formula of* $\mathcal{L}_{\mathfrak{I},Q}$ *which satisfies the condition of Theorem 4. (Again we write '*$\mathsf{L}\varphi$*' for* $[\lambda]\,\varphi/q_1$*.) Let* $\mathcal{A}$ *be an axiom system for* $\mathcal{L}_{\mathfrak{I},Q}$ *which is strongly* $\mathcal{K}$*-consistent in* $\mathcal{L}_{\mathfrak{I},Q}(\mathsf{N})$ *and strongly* $\mathcal{K}$*-complete in* $\mathcal{L}_{\mathfrak{I},Q}$*. Then* $\mathcal{A}'$ *is strongly* $\mathcal{K}$*-complete in* $\mathcal{L}_{\mathfrak{I},Q}(\mathsf{N})$*.*

The proof of theorem 5 has the irritating—if common—property that when you try to write it up either you don't write enough, or else you write far too much. I have chosen the second alternative. However, since the idea behind the proof is very simple, I will first give a rough description of it. He who is satisfied with this informal explanation may save himself the trouble of reading the proof itself.

To show that a consistent set Δ of sentences of $\mathcal{L}_{\mathfrak{I},Q}(\mathsf{N})$ has an interpretation in which all the sentences of Δ are true simultaneously, we proceed as if the subformulae of those sentences that begin with N were atomic formulae. In this way we can regard Δ as a set of sentences of a language $\mathcal{L}_{\mathfrak{I},Q'}$ (where $Q \subseteq Q'$) for which, since it is a consistent set relative to $\mathcal{A}$, there is an interpretation in which all sentences are simultaneously true. This interpretation gives us, in a completely straightforward way, an interpretation in which, at one moment, all the members of Δ are true according to the truth definition for $\mathcal{L}_{\mathcal{I},Q}(\mathrm{N})$.

PROOF. Let us call a formula φ of $\mathcal{L}_{\mathcal{I},Q}(\mathrm{N})$ an *n-place irreducible predicate* if (1) for each i such that $1 \leq i \leq n$ there is exactly one free occurrence in φ of the variable v_i; (2) for $i = 1, \ldots, n-1$ the free occurrence of v_{i+1} in φ is the first free occurrence in φ to the right of the free occurrence of v_i in φ; (3) no other variables have free occurrences in φ; and (4) φ contains no individual constants. For any n-place irreducible predicate φ and individual symbols (constants or variables) $\alpha_1, \ldots, \alpha_n$ we write $\varphi(\alpha_1 \ldots \alpha_n)$ for $[\varphi]\ \alpha_1/v_1, \ldots, \alpha_n/v_n$. Let G be a function which assigns to each n-place irreducible predicate of $\mathcal{L}_{\mathcal{I},Q}(\mathrm{N})$ a different n-place predicate symbol not in Q. Let $Q' = Q \cup \text{Range G}$. If ψ is any formula of $\mathcal{L}_{\mathcal{I},Q}$ and n is the total number of occurrences which are either occurrences of individual constants or free occurrences of variables, then there is a unique n-place irreducible predicate φ of $\mathcal{L}_{\mathcal{I},Q}$ such that $\psi = \varphi\ (\alpha_1 \ldots \alpha_n)$, where each α_i is a variable or individual constant. The function G therefore induces a function G′ from the formula of $\mathcal{L}_{\mathcal{I},Q}$ to atomic formulae of $\mathcal{L}_{\mathcal{I},Q'}$ defined by the condition

$$G'(\psi) = G(\varphi)\ (\alpha_1 \ldots \alpha_n),$$

where φ, n are such that φ is an n-place irreducible predicate and $\psi = \varphi\ (\alpha_1 \ldots \alpha_n)$. The function G′ determines in turn a function G″ from the formulae of $\mathcal{L}_{\mathcal{I},Q}(\mathrm{N})$ to formulae of $\mathcal{L}_{\mathcal{I},Q'}$, defined recursively by:

(i) if $\psi \in \mathcal{L}_{\mathcal{I},Q}$ then $G''(\psi) = \psi$.
(ii) $G''(\mathrm{N}\psi) = G'(\psi)$.
(iii) $G''(\mathrm{C}_i^n(\psi_1 \ldots \psi_n)) = \mathrm{C}_i^n(G''(\psi_1) \ldots G''(\psi_n))$;
$G''((\forall v_i)\psi) = (\forall v_i)G''(\psi)$;
$G''((\exists v_i)\psi) = (\exists v_i)G''(\psi)$.

Now let Δ be a set of sentences of $\mathcal{L}_{\mathcal{I},Q}(\mathrm{N})$ which is consistent relative to $\mathcal{A}'$. Let Δ' be the union of Δ and the set of all universal closures of formulae of the forms $\varphi \leftrightarrow \mathsf{N}\varphi$ and $\mathsf{L}(\mathsf{N}\varphi \rightarrow \mathsf{LN}\varphi)$, where φ is a formula of $\mathcal{L}_{\mathcal{I}, Q}(\mathsf{N})$. Then Δ' is consistent relative to $\mathcal{A}'$, since all the added formulae are soundly provable from $\mathcal{A}'$. Clearly $G''(\Delta')$ is a set of formulae of $\mathcal{L}_{\mathcal{I}, Q'}$ which is consistent relative to $\mathcal{A}$. Since $\mathcal{A}$ is strongly $\mathcal{K}$-complete in $\mathcal{L}_{\mathcal{I}, Q'}$ there is a $\mathcal{J} \in \mathcal{K}$ and an interpretation $\mathcal{M} = \langle U, R, t_0 \rangle$ relative to $\mathcal{J}$ such that for all $\varphi \in G''(\Delta)$, $[\varphi]^{\mathcal{I}, \mathcal{J}}_{\mathcal{M}, t} = 1$. Let H be a function which assigns to each constant function from $\mathcal{J}$ to an element of U a different individual constant which is not in Q. Let $\mathcal{M}' = \langle U, R, H, t_0 \rangle$; $Q'' = Q \cup \text{Dom } H$; $Q''' = Q' \cup Q''$. $\mathcal{M}'$ is a referentially complete interpretation for $\mathcal{L}_{\mathcal{I}, Q''}$, relative to $\mathcal{J}$. For $t \in T$, let Γ_t be the set of all sentences φ of $\mathcal{L}_{\mathcal{I}, Q'''}$ such that $[\varphi]^{\mathcal{I}, \mathcal{J}}_{\langle \mathcal{M}', t_0 \rangle} = 1$. The function G'' can be extended in the obvious manner to a function from the formulae of $\mathcal{L}_{\mathcal{I}, Q''}(\mathsf{N})$ to formulae of $\mathcal{L}_{\mathcal{I}, Q'''}$. We will refer to this extension also as G''. Let R' be the restriction of R to Q''. Then $\mathcal{M}'' = \langle U, R,' H, t_0 \rangle$ is a referentially complete interpretation for $\mathcal{L}_{\mathcal{I}, Q''}$ relative to $\mathcal{J}$. I claim that:

(1) for every sentence $\varphi \in \Delta$, $[\varphi]^{\mathcal{I}, \mathcal{J}}_{\mathcal{M}'', t_0} = 1$.

Since $\Delta \subseteq (G'')^{-1} (\Gamma_{t_0})$, (1) follows from

(2) for every sentence φ of $\mathcal{L}_{\mathcal{I}, Q'''}$ and $t \in T$

$$[\varphi]^{\mathcal{I}, \mathcal{J}}_{\mathcal{M}'', t} = 1 \text{ iff } G''(\varphi) \in \Gamma_t.$$

(2) is proved by induction on sentences of $\mathcal{L}_{\mathcal{I}, Q'''}$. It is clear that (2) holds for atomic sentences, that if (2) holds for $\varphi_1, \ldots, \varphi_n$ then it holds for $\mathsf{C}^n_i(\varphi_1 \ldots \varphi_n)$, and that if (2) holds for φ then it holds for $(\forall v_i)$ $[\varphi]$ v_i/a and for $(\exists v_i)$ $[\varphi]$ v_i/a. Suppose that $\varphi = \mathsf{N}\psi$ and that (2) holds for ψ. First assume that $[\varphi]^{\mathcal{I}, \mathcal{J}}_{\mathcal{M}'',} = 1$. Then $[\psi]^{\mathcal{I}, \mathcal{J}}_{\mathcal{M}'', t_0} = 1$. So by induction hypothesis $G''(\psi) \in \Gamma_{t_0}$. Let ψ' be a formula of $\mathcal{L}_{\mathcal{I}, Q}(\mathsf{N})$ such that for some $v_{i_1}, \ldots, v_{i_k}$ and constants $a_1, \ldots, a_n$, $\psi = [\psi'] \, v_{i_1}/a_1, \ldots, v_{i_k}/a_k$. Δ' contains the formula $(\forall v_{i_1}) \ldots (\forall v_{i_k})$ $(\psi' \leftrightarrow \mathsf{N}\psi')$. Therefore Γ_{t_0} contains $G''((\forall v_{i_1}) \ldots (\forall v_{i_k}) (\psi' \leftrightarrow \mathsf{N}\psi'))$, which is equal to $(\forall v_{i_1}) \ldots (\forall v_{i_k}) (G''(\psi') \leftrightarrow G''(\mathsf{N}\psi'))$. Then Γ_{t_0} also contains $[G''(\psi') \leftrightarrow G''(\mathsf{N}\psi')] \, a_1/v_{i_1}, \ldots, a_k/v_{i_k}$, which is $G''(\psi) \leftrightarrow G''(\mathsf{N}\psi)$. So $G''(\mathsf{N}\psi) \in \Gamma_{t_0}$. By an analogous argument we can infer that $\mathsf{L}G''(\mathsf{N}\psi) = G''(\mathsf{LN}\psi) \in \Gamma_{t_0}$. It follows that $G''(\mathsf{N}\psi) \in \Gamma_t$.

In a similar fashion one shows that if $G''(\varphi) \in \Gamma_t$ then $[\varphi]^{\mathcal{J},\mathcal{T}}_{\mathcal{M}'',t} = 1$. It follows that $\mathcal{A}^*$ is strongly $\mathcal{K}$-complete in $\mathfrak{L}_{\mathcal{J},Q}(\mathsf{N})$.

It is clear that Theorem 5 applies to the language $\mathfrak{L}_{1,Q}$ when $\mathcal{K}$ is a class of *linear* orderings.

§ 5.

In this section I will show that in general there is not for every sentence φ of $\mathfrak{L}_{\mathcal{J},Q}(\mathsf{N})$ a sentence ψ of $\mathfrak{L}_{\mathcal{J},Q}$ such that $\varphi \leftrightarrow \psi$ is valid. To this purpose I will give two theorems, which together will show that the elimination of N (in the sense of Theorem 2) is indeed impossible for most reasonable choices of $\mathcal{J}$, Q, and $\mathcal{K}$. Undoubtedly similar theorems could be obtained for combinations of $\mathcal{J}$, Q, and $\mathcal{K}$ which are not covered by the theorems presented here. But it seems to me that these two give a sufficiently clear indication of along what lines such results can be proved.

The first theorem is concerned with a large class of tenses, viz., all those which are *invariant*, according to the following definition.

DEFINITION 25. An n-place tense $\mathcal{J}$ is *$\mathcal{K}$-invariant* iff whenever $\mathcal{T}, \mathcal{T}' \in \mathcal{K}$, g is an order-isomorphism from $\mathcal{T}$ to $\mathcal{T}'$, and $p_1, \ldots, p_n \in \{0, 1\}^T$, then $\mathcal{J}(\mathcal{T}')(g(p_1), \ldots, g(p_n)) = g(\mathcal{J}(\mathcal{T})\ (p_1, \ldots, p_n))$, where by $g(p_i)$ we understand that function $\{0, 1\}^{T'}$ such that for any $t \in T'$, $g(p_i)\ (t) = p_i(g^{-1}(t))$.

It is my opinion that any intuitively plausible tense operator ought to have an invariant tense for its meaning. Thus the restriction to invariant tenses is from an intuitive point of view no essential limitation at all.

The time structures that the first theorem considers are the dense linear orderings without endpoints. Let $\mathcal{K}_0$ be the class of these orderings. Let $\mathcal{J}_0$ be the class of all $\mathcal{K}_0$-invariant tenses. I will construct, for an arbitrary set of predicate symbols and individual constants which contains at least one predicate symbol, a referentially complete interpretation $\mathcal{M} = \langle U, R, H, 0\rangle$ for $\mathfrak{L}_{\mathcal{J}_0,Q}(\mathsf{N})$, relative to some member $\mathcal{T}$ of $\mathcal{K}_0$, such that (i) if

$\mathcal{L}_{\mathcal{T}_0, Q}$ contains a sentence λ satisfying the condition of Theorem 4 then there is a sentence ψ of $\mathcal{L}_{\mathcal{T}_0, Q}(\mathsf{N})$ which is true in $\mathcal{M}$ only at 0; and (ii) every sentence φ of $\mathcal{L}_{\mathcal{T}_0, Q}$ is either true at no point, or else at infinitely many. This will show that it is impossible to eliminate N from ψ in the sense of Theorem 2.

For simplicity I will assume that Q contains only one element, the 1-place predicate letter Q^1_1. It can be seen from the construction below—and will be argued after the construction has been given—that this assumption is not essential.

Let $Q' = Q \cup \{c_i : i \in \omega\}$. T will be equal to $T' \cup \{0\}$, where T' is a set of rational numbers different from 0, $\mathcal{M}$ the quadruple $\langle \omega, R, H, 0\rangle$; for every $t \in T$, $R(Q^1_1)(t)$ will be an infinite, coinfinite subset S'_t of ω,[13] such that (a) for every $t \in T$ there are $t' < t$ and $t'' > t$ such that $S_{t'} \cap S_t = S_{t''} \cap S_t = \emptyset$; and (b) for every $t \in T$, $S_t \cap S_0 \neq \emptyset$. H will be a $1-1$ function mapping each c_i onto the (constant) function from T to $\{i\}$. In view of (a), (b) we have that

(1) the sentence $\mathsf{L}(\exists v_0)(Q^1_1(v_0) \wedge \mathsf{N}Q^1_1(v_0))$ is true in $\mathcal{M}$ at 0 but at no other moment.

Moreover, the sets S_t for $t \in T'$ will be defined in such a way that

(2) each sentence of $\mathcal{L}_{\mathcal{T}_0, Q}$ is true either at all $t \in T$, or else at no t.

In order to describe the construction I will first make a few remarks on a certain kind of Boolean algebra, whose properties will play an important role in that construction, as well as in the argument which follows it.

Let U be an infinite set and $\mathcal{U}$ a subset of $\mathcal{P}(U)$. For any finite subsets K, L of U let $\mathcal{U}(K, L)$ be the set of all members V of $\mathcal{U}$ such that $K \subseteq V$ and $L \cap V = \emptyset$. Let $\mathcal{B}(U, \mathcal{U})$ be the Boolean algebra generated by the sets $\mathcal{U}(K, L)$ under the operations of finite set theoretic union, intersection, and complementation relative to $\mathcal{P}(U)$. One easily verifies that $\mathcal{B}(U, \mathcal{U})$ is generated by the sets of the forms $\mathcal{U}(\{u\}, \emptyset)$ and $\mathcal{U}(\emptyset, \{u\})$, and consequently that for every element V of $\mathcal{B}(U, \mathcal{U})$ there is a number $k \leq 1$ and a function B such that Dom $B = \{1, \ldots, k\}$ and for $i = 1, \ldots, k$ B_i

[13] A subset K of ω is *coinfinite* iff $\omega - K$ is infinite.

is a pair of finite subsets B_i^1, B_i^2 of U and $V = \bigcup_{i=1}^k \mathcal{U}(B_i^1, B_i^2)$. Such a function B will be called a *representation* of the element V. I will write $\mathcal{U}(B)$ for $\bigcup_{i=1}^k \mathcal{U}(B_i^1, B_i^2)$.

Now let S be the set of all infinite, coinfinite subsets of ω. Then, whenever K, L are finite subsets of ω, and $K \cap L = \emptyset$ then $S(K, L) \neq \emptyset$; and thus every representation B such that for some $i \in \text{Dom } B$, $B_i^1 \cap B_i^2 = \emptyset$ represents an element of $\mathcal{B}(\omega, S)$ different from $\emptyset$. Exactly the same properties hold for the algebra $\mathcal{B}(V, \mathcal{V})$ where V is the set $\{v_0, v_1, \ldots\}$ and $\mathcal{V}$ is the set of all infinite, coinfinite subsets of V. The algebras $\mathcal{B}(\omega, S)$ and $\mathcal{B}(V, \mathcal{V})$ are of course isomorphic. In fact, let us understand by a *perfect assignment* a $1-1$ function from V onto ω. Then for any perfect assignment h and finite subsets K, L of V $\{h(W): W \in \mathcal{V}(K, L)\} = S(h(K), h(L))$— so that h induces an isomorphism between $\mathcal{B}(V, \mathcal{V})$ and $\mathcal{B}(\omega, S)$.

T' and the sets S_t for $t \in T'$ are constructed as follows. Let *Rat* be the set of rational numbers, and let $<$ be the natural ordering on *Rat*. Let $\{r_i\}_{i \in \omega}$ be an enumeration of $Rat - \{0\}$ without repetitions. Let $\{J_k\}_{k \in \omega}$ be an enumeration of all pairs of mutually disjoint finite subsets of ω. Let W be an enumeration of all pairs $\langle i, -1\rangle$ and $\langle i, +1\rangle$ and all triples $\langle i, j, k\rangle$ where $i, j, k \in \omega$ and $i \neq j$. Let $\{E_a\}$ be an enumeration of all members of S.

For $n = 0, 1, 2, \ldots$ construct a pair $\langle i_n, S_n\rangle$ consisting of a natural number i_n and a member S_n of S, and a subsequence W_n of W as follows:

(i) $\langle i_0, S_0\rangle = \langle 1, E_0\rangle$; $W_0 = W$.
(ii) Suppose that for $m \leq n$ $\langle i_m, S_m\rangle$ and W_m have been constructed.

Let w be the first element of W_n such that either

(a) $w = \langle i, +1\rangle$ and $i = i_m$ for some $m \leq n$; or
(b) $w = \langle i, -1\rangle$ and $i = i_m$ for some $m \leq n$; or
(c) $w = \langle i, j, k\rangle$, $i = i_m$ and $j = i_{m'}$ for some $m, m' \leq n$.

(It will be clear from the construction that W_n does indeed contain such an element w). In case (a) let i_{n+1} be the first natural number different from $i_0, \ldots, i_n$ such that $r_{i_{n+1}}$ is a rational number $> r_{i_m}$, and let S_{n+1} be the first member of $\{E_a\}$ different from

$S_0, \ldots, S_n$ which is disjoint from S_m. In case (b) let i_{n+1} be the first number different from $i_0, \ldots, i_n$ such that $r_{i_{n+1}}$ is a negative rational number $< r_{i_m}$; let S_{n+1} be as under (a). In case (c) let i_{n+1} be the first number different from $i_0, \ldots, i_n$ such that $r_{i_{n+1}}$ is between r_{i_m} and $r_{i_{m'}}$ and let S_{n+1} be the first member of $\{E_a\}$ different from $S_0, \ldots, S_n$ which belongs to $S(J_k)$. In all cases (a), (b), (c) let W_{n+1} be the sequence which we obtain when we eliminate w from W_n. Let $T = \{r_{i_n}: n \in \omega\} \cup \{0\}$, and let $\mathfrak{T} = \langle T, < \rangle$, where $<$ is the natural ordering of T. For $t \in T$, $t \neq 0$ let $S_t = S_n$, where $t = r_{i_n}$; and let S_0 be a member of S which has at least one member in common with each S_t where $t \in T$, $t \neq 0$. We show that the interpretation $\langle \omega, R, H, 0 \rangle$ has indeed the properties (1) and (2). That (1) holds is clear. It is somewhat less obvious that (2) is true. The next few pages will be concerned with showing that this is so.

Let $n_0, \ldots, n_k$ be a sequence of distinct natural numbers. For $K \subseteq \{v_0, \ldots, v_k\}$ we understand by '$K(n_0, \ldots, n_k)$' the set $h(K)$, where h is any perfect assignment such that for $i = 1, \ldots, k$, $h(v_i) = n_i$; for any representation B in the range of which occur only subsets of $\{v_0, \ldots, v_k\}$ understand by $B(n_0, \ldots, n_k)$ the function which assigns to each $i \in \mathrm{Dom}\, B$ the pair $\langle B_i^1(n_0, \ldots, n_k), B_i^2(n_0, \ldots, n_k) \rangle$; for any formula φ of $\mathfrak{L}_{\mathfrak{I}_0, Q}$, all free variables of which are among $v_0, \ldots, v_k$, let $\varphi(n_0, \ldots, n_k)$ be the formula $[\varphi]\, c_{n_0}/v_0, \ldots, c_{n_k}/v_k$. For any sentence φ of $\mathfrak{L}_{\mathfrak{I}_0, Q}$, let $T(\varphi)$ be the set of all $t \in T$ such that $[\varphi]^{\mathfrak{I}, \mathfrak{T}}_{\mathfrak{M}, t} = 1$. For any representation B of an element of $\mathbf{B}(\omega, S)$ let $T(B) = \{t \in T: S_t \in S(B)\}$.

It is a well-known fact of ordinary predicate logic with identity that for every formula φ there is a logically equivalent formula φ' satisfying the condition that

(3) every subformula of φ' which begins with a quantifier is either of the form
$(\exists v_i)(\sim v_i = v_{i_1} \wedge \ldots \wedge \sim v_i = v_{i_n} \wedge \psi)$ or of the form
$(\forall v_i)\ (\sim v_i = v_{i_1} \wedge \ldots \wedge \sim v_i = v_{i_n} \rightarrow \psi)$, where $v_{i_1}, \ldots, v_{i_n}$ are all the free variables of ψ.

One easily verifies that this is also true for the languages $\mathfrak{L}_{\mathfrak{I}, Q}$ and $\mathfrak{L}_{\mathfrak{I}, Q}(\mathsf{N})$, in the sense that for any formula φ of $\mathfrak{L}_{\mathfrak{I}, Q}$ or $\mathfrak{L}_{\mathfrak{I}, Q}(\mathsf{N})$ there is a formula φ' satisfying condition (3) such that

$\varphi \leftrightarrow \varphi'$ is $\mathcal{K}$-valid (whatever $\mathcal{K}$ may be!). Formulae which satisfy (3) I will call *restricted*.

I will show that

(4) for each formula φ of $\mathcal{L}_{\mathfrak{I},Q}$ the free variables of which are among $\{v_0, \ldots, v_k\}$ there is a representation $B(\varphi)$ in the range of which occur only subsets of $\{v_0, \ldots, v_k\}$ such that for any choice of distinct natural numbers $n_0, \ldots, n_k$, $T(\varphi\,(n_0, \ldots, n_k) = T(B(\varphi)(n_0, \ldots, n_k))$; either $B(\varphi)$ is $\{\langle 1, \langle\{v_0\}, \{v_0\}\rangle\rangle\}$, or else for each $i \in \mathrm{Dom}\, B$, $B_i^1 \cap B_i^2 = \emptyset$.

I will prove (4) by induction on the complexity of restricted formulae. In view of the equivalence of each formula with a restricted formula this will establish (4) for all formulae of $\mathcal{L}_{\mathfrak{I}_0, Q'}$.

If φ is the formula q_i put $B(\varphi) = \{\langle 1, \langle\{v_0\}, \{v_0\}\rangle\rangle\}$; if φ is the formula $v_i = v_i$, put $B(\varphi) = \{\langle 1, \langle\{v_i\}, \emptyset\rangle\rangle, \langle 2, \langle\emptyset, \{v_i\}\rangle\rangle\}$; if φ is the formula $v_i = v_j$, where $i \neq j$, put $B(\varphi) = \{\langle 1, \langle\{v_0\}, \{v_0\}\rangle\rangle\}$; if φ is the formula $Q_1^1(v_i)$, put $B(\varphi) = \{\langle 1, \langle\{v_i\}, \emptyset\rangle\rangle\}$.

Suppose that $\varphi = \mathrm{C}_i^n(\varphi_1 \ldots \varphi_n)$ and that for $i = 1, \ldots, n$ $B(\varphi)$ has been determined. Let $\varrho_1, \ldots, \varrho_{2^n}$ be all formulae of the form $\psi_1 \wedge \ldots \wedge \psi_n$, where for $i = 1, \ldots, n$ ψ_i is φ_i or ψ_i is $\sim\varphi_i$. It follows from our remarks on the Boolean algebras $\mathcal{B}(\omega, S)$ and $\mathcal{B}(V, \mathcal{V})$ that there are, for $s = 1, \ldots, 2^n$, representations B_s such that for any distinct natural numbers $n_0, \ldots, n_k$, $T(\varrho_s(n_0, \ldots, n_k)) = T(B_s(n_0, \ldots, n_k))$. Thus for each s it is the case that either for all choices of distinct numbers $n_0, \ldots, n_k$ $T(\varrho_s(n, \ldots, n_k))$ is empty, or else for all such choices $T(\varrho_s(n_0, \ldots, n_k))$ is dense in $\mathfrak{T}$. Moreover, the non-empty sets $T(\varrho_s(n_0, \ldots, n_k))$ form, for any such choice, a partition of T.

Now let $t, t' \in T$ and let $n_0, \ldots, n_k$, and $m_0, \ldots, m_k$ be two choices of k distinct natural numbers such that for some $s \leq 2^n$ $t \in T(\varrho_s(n_0, \ldots, n_k))$ and $t' \in T(\varrho_s(m_0, \ldots, m_k))$. Then ,by a simple generalization of Cantor's argument, there is an order-automorphism g of $\mathfrak{T}$ such that $g(t) = t'$ and for $i = 1, \ldots, n$ $g(T(\varphi_i(n_0, \ldots, n_k))) = g(\varphi_i(m_0, \ldots, m_k))$. Since $\mathfrak{I}_0(i)$ is $\{\mathfrak{T}\}$-invariant it follows that $t \in T((\mathrm{C}_i^n(\varphi_1, \ldots, \varphi_n))(n_0, \ldots, n_k))$ iff $t' \in T((\mathrm{C}_i^n(\varphi_1 \ldots \varphi_n))(m_0, \ldots, m_k))$. Let $s_1, \ldots, s_u$ be all those numbers $s \leq 2^n$ such that for any distinct $n_0, \ldots, n_k$, $T(\varrho_s(n_0, \ldots, n_k)) \subseteq T((\mathrm{C}_i^n(\varphi_1 \ldots \varphi_n))(n_0, \ldots,$

$n_k)$). Now let $B(\varphi)$ be a representation the range of which is the union of the ranges of $B_{s_1}, \ldots, B_{s_u}$. Then $B(\varphi)$ stands in the by (4) required relation to φ.

Suppose that φ is a formula beginning with an existential quantifier, of which all free variables are among $v_0, \ldots, v_k$. For convenience let us assume that φ is the formula

$$(\exists v_{k+1})\,(\sim v_{k+1} = v_0 \wedge \ldots \wedge \sim v_{k+1} = v_k \wedge \psi).$$

Suppose first that $B(\psi) = \{\langle 1, \langle \{v_0\}, \{v_0\}\rangle\rangle\}$. Then for any choice of distinct $n_0, \ldots, n_k, n_{k+1}$, $T(\psi(n_0, \ldots, n_k, n_{k+1})) = \emptyset$. So $T(\varphi(n_0, \ldots, n_k)) = \emptyset$ for all distinct $n_0, \ldots, n_k$. So we may put $B(\varphi) = \{\langle 1, \langle \{v_0\}, \{v_0\}\rangle\rangle\}$. Now suppose instead that $B(\psi) \neq \{\langle 1, \langle \{v_0\}, \{v_0\}\rangle\rangle\}$. Assume that Dom $B(\psi) = \{1, \ldots, p\}$. For each $i \leq p$ let $C_i^1 = B_i^1 - \{v_{k+1}\}$, $C_i^2 = B_i^2 - \{v_{k+1}\}$. I claim that

(5) for any choice of distinct $n_0, \ldots, n_k$
$$T(\varphi(n_0, \ldots, n_k)) = \bigcup_{i \leq p} S(C_i^1(n_0, \ldots, n_k), C_i^2(n_0, \ldots, n_k)).$$

To show (5) first suppose that $t \in T(\varphi\,(n_0, \ldots, n_k))$. Then there is a number $n_{k+1} \neq n_0, \ldots, n_k$ such that $t \in T(\psi(n_0, \ldots, n_k, n_{k+1}))$. So $t \in S(B_i^1(n_0, \ldots, n_k, n_{k+1}), B_i^2(n_0, \ldots, n_k, n_{k+1}))$ for some $i \leq p$. So $B_i^1(n_0, \ldots, n_k, n_{k+1}) \subseteq S_t$ and $B_i^2(n_0, \ldots, n_k, n_{k+1}) \cap S_t = \emptyset$. So $C_i^1(n_0, \ldots, n_k) \subseteq S_t$ and $C_i^2(n_0, \ldots, n_k) \cap S_t = \emptyset$. So $t \in \bigcup_{i \leq p} S(C_i^1(n_0, \ldots, n_k), C_i^1(n, \ldots, n_k))$. Now suppose that $t \in S(C_i^1(n_0, \ldots, n_k), C_i^2(n_0, \ldots, n_k))$ for some $i \leq p$. Since $B_i^1 \cap B_i^2 = \emptyset$, $v_{k+1} \notin B_i^1$ or $v_{k+1} \notin B_i^2$. Suppose that $v_{k+1} \notin B_i^1$. Let n_{k+1} be a number not in $S_t \cup \{n_0, \ldots, n_k\}$ (such a number exists as S_t is coinfinite). Then $B_i^1(n_0, \ldots, n_k, n_{k+1}) = C_i^1(n_0, \ldots, n_k)$ and $B_i^2(n_0, \ldots, n_k, n_{k+1}) \cap S_t = \emptyset$. So $S_t \in S(B_i^1(n_0, \ldots, n_k, n_{k+1}), B_i^2(n_0, \ldots, n_k, n_{k+1}))$. So $t \in T(\psi\,(n_0, \ldots, n_k, n_{k+1})) \subseteq T(\varphi(n_0, \ldots, n_k))$. The case where $v_{k+1} \notin B_i^2$ is treated similarly. Now let $B(\varphi) = \{\langle i, \langle C_i^1, C_i^2\rangle\colon i = 1, \ldots, p\}$. Then $B(\varphi)$ stands in the required relation to φ. This completes the proof of (5). The case where φ begins with a universal, rather than an existential, quantifier is treated in exactly the same way. This completes the proof of (4).

Every set $T(B(\varphi)\,(n_0, \ldots, n_k))$ is either empty or else dense in $\mathcal{J}$; it follows from (4) that this is true also of the set of all $t \in T$ where a given sentence of $\mathfrak{L}_{\mathfrak{J}_0, a'}$ is true in $\mathfrak{M}$. In particular, since for every pair of disjoint finite subsets K, L of $\{v_0, \ldots, v_k\}$ and every $t \in T$ there are distinct numbers $n_0, \ldots, n_k$ such that $S_t \in S(K(n_0, \ldots, n_k), L(n_0, \ldots, n_k))$ a sentence of $\mathcal{L}_{\mathcal{J}_0, Q'}$ will either be true in $\mathcal{M}$ at all t, or else false at all t.

The assumption, made earlier, that Q' contains only the predicate letter Q^1_1, can be considerably weakened. In fact, all we have to assume about Q is that it contains at least one predicate letter, of one or more places. That this assumption suffices can be seen as follows. Let us first assume that Q contains only predicate symbols. For any predicate letter $Q^n_i \in Q$ let the interpretation $R(Q^n_i)$ in the constructed interpretation $\mathcal{M}$ be defined by $R(Q^n_i)(t) = \{\langle m_0, \ldots, m_n\rangle : m_0 \in S_t$ and $m_0 = m_1 = \ldots = m_n\}$. Then clearly the sentence $\mathsf{L}(\exists v_0)\,(Q^n_i(v_0 \ldots v_0) \wedge \mathsf{N}Q^n_i(v_0 \ldots v_0))$ is true only at 0. Associate with every formula $Q^n_i(v_{i_1} \ldots v_{i_n})$ where for some r, $s \leq n$, $v_{i_r} \neq v_{i_s}$, the representation $\{\langle 1, \langle\{v_0\}, \{v_0\}\rangle\rangle\}$ and with every formula $Q^n_i(v_i \ldots v_i)$ the representation $\{\langle 1, \langle\{v_i\}, \emptyset\rangle\rangle\}$. One easily verifies that in either case the representation stands in the by (4) required relation to the formula. This, together with the inductive steps proved above, establishes that (4) holds for all formulae of $\mathcal{L}_{\mathcal{J}_0, Q'}$ so that sentences of $\mathcal{L}_{\mathcal{J}_0, Q'}$ are again either true nowhere or else at a set of moments which is dense in $\mathcal{J}$.

Now suppose that Q contains individual constants as well. Let c_β be one of those. Let $\mathcal{J}$ be as before. The interpretation $\mathcal{M}$ will now have the form $\langle \omega \cup \{\omega\}, R, H', 0\rangle$, where $R(Q^n_i)$ is defined as before; $R(c_\beta)\,(t) = \omega$ for all $c_\alpha \in Q$ and $t \in T$; and H' is a $1-1$ function from a set of constants not in Q onto the set of constant functions from T into $\omega \cup \{\omega\}$. We can again prove (4) by induction, now restricting attention to formulae in which every subformula beginning with $(\exists v_i)$ or $(\forall v_i)$ is of the form $(\exists v_i)\,(\sim v_i = v_{i_1} \wedge \ldots \wedge \sim v_i = v_{i_k} \wedge \sim v_i = c_\beta \wedge \psi)$, or $(\forall v_i)\,(\sim v_i = v_{i_1} \wedge \ldots \wedge \sim v_i = v_{i_k} \wedge \sim v_1 = c_\beta \rightarrow \psi)$, respectively; and associating with every atomic formula $Q^n_i(\beta_1, \ldots, \beta_n)$, where at least one β_i is a constant in Q, the representation $\{\langle 1, \langle\{v_0\}, \{v_0\}\rangle\rangle\}$.

The above results can be summarized in the following

THEOREM 6. *Let $\mathcal{K}$ be a class of partial orderings which contains at least one dense linear ordering without endpoints. Let $\mathcal{J}$ be an* *indexed family of tenses which are $\mathcal{K}$-invariant. Let Q be any set of predicate letters and individual constants which contains at least one predicate letter. Then there is a sentence ψ of $\mathcal{L}_{\mathcal{T}, Q}(\mathsf{N})$ such that for no sentence φ of $\mathcal{L}_{\mathcal{T}, Q}$ is $\psi \leftrightarrow \varphi$ $\mathcal{K}$-valid.*

The second theorem of this section is concerned with discrete orderings. I will construct an interpretation relative to the integers (regarded as a linearly ordered structure) in which again a sentence containing N is true only at 0, while all sentences not containing N will be true either nowhere or else at arbitrarily large positive and negative integers. This result will not be as general as Theorem 6, however, since it will only apply to languages $\mathcal{L}_{1, Q}$ and $\mathcal{L}_{1, Q}(\mathsf{N})$. For the construction I will assume that $Q = \{Q_1^1\}$. As in the previous proof this assumption can be weakened to the hypothesis that Q contains at least one predicate letter.

Let T' be the set of all integers different from 0, and let $T = T' \cup \{0\}$. The interpretation $\mathcal{M}$ will again be of the form $\langle \omega, R, H, 0 \rangle$, where H is as before and $R(Q_1^1)(t)$ is a member S_t of S. I will now describe the construction of the S_t. The idea is to construct successively S_1, S_{-1}, S_2, S_{-2}, etc., and finally S_0. Let $\{E_a\}_{a<2^\omega}$ and $\{J_k\}_{k<\omega}$ be the enumerations used in the previous proof. Let $\{i_j\}_{j<\omega}$ be the enumeration of the integers different from 0 which starts with 1, -1, 2, $-2, \ldots$ and continues in the obvious way. Let W be an enumeration of all pairs $\langle i, -1 \rangle$ and $\langle i, +1 \rangle$, where i is an integer $\neq 0$, and all pairs $\langle k, -2 \rangle$ and $\langle k, +2 \rangle$, where k is a natural number. I construct for $n = 0, 1, 2, \ldots$ infinite, coinfinite subsets S'_n of ω and subsequences W_n of W as follows:

(1) Let $S'_0 = E_0$, $W_0 = W$.

(2) Suppose that S'_m, W_m have been constructed for $m \leq n$. (a) Suppose that i_{n+1} is positive. Let w be the first member of W_n such that either $w = \langle i, +1 \rangle$ and $i = i_m$ for some $m \leq n$; or $w = \langle k, +2 \rangle$. If $w = \langle i, +1 \rangle$ let S'_{n+1} be the first E_a not yet used which is disjoint from S'_m; if $w = \langle k, +2 \rangle$ let S'_{n+1} be the first E_a not yet used which belongs to $S(J_k)$. (b) Suppose that i_{n+1} is negative. Let w be the first member of W_n such that either $w = \langle i, -1 \rangle$ and $i = i_m$ for some $m \leq n$; or $w = \langle k, -2 \rangle$; if $w = \langle i, -1 \rangle$ let S'_{n+1} be the first E_a not yet used which is disjoint from S'_m; if $w = \langle k, -2 \rangle$

let S'_{n+1} be the first E_a not yet used which belongs to $S(J_k)$. In each of the cases under (a) and (b) let W_{n+1} be the sequence which results if we omit w from W_n. Let, for each $t \in T'$, $S_t = S'_n$, where $t = i_n$. Let S_0 be an infinite, coinfinite subset of ω which contains an element from each S_t such that $t \in T'$. One easily verifies that (i) for each $t \in T'$ there are $t' < t$ and $t'' > t$ such that $S_{t'} \cap S_t = S_{t''} \cap S_t = \emptyset$; and (ii) for any disjoint finite subsets K, L of ω and $t \in T$ there are $t' < t$ and $t'' > t$ such that $S_{t'}$ and $S_{t''}$ belong to $S(K, L)$. From (i) follows that the sentence $\mathsf{L}(\exists v_0)(Q_1^1(v_0) \wedge \mathsf{N}Q_1^1(v_0))$ is true in $\mathfrak{M}$ only at 0. (ii) enables us to prove that condition (4) holds (for formulae of $\mathfrak{L}_{1,Q}$). Again the proof is by induction on restricted formulae. It proceeds along the same lines as the argument given above, the only difference being that the inductive step where φ is of the form $\mathsf{C}_i^n(\varphi_1 \ldots \varphi_n)$ is now replaced by the two special cases where φ is $\mathsf{G}\psi$ or $\mathsf{H}\psi$. I will consider the case where φ is $\mathsf{G}\psi$. Suppose that φ is $\mathsf{G}\psi$ and that $B(\psi)$ has been defined. If $\mathfrak{V}(B(\psi))$ consists of all infinite, coinfinite subsets of V then for every choice of distinct natural numbers $n_0, \ldots, n_k$ (where I assume that all sets which occur in the range of $B(\psi)$ are included in $\{v_0, \ldots, v_k\}$), and every $t \in T$ $\psi(n_0, \ldots, n_k)$ is true at t. So $\mathsf{G}\psi(n_0, \ldots, n_k)$ is true at all t, and we may therefore put $B(\varphi) = \{\langle 1, \langle \emptyset, \emptyset \rangle \rangle\}$. Now suppose that $\mathfrak{V}(B(\psi))$ does not consist of all infinite, coinfinite sets of variables. Then, by our remarks on the Boolean algebras $\mathfrak{B}(\omega, S)$ and $\mathfrak{B}(V, \mathfrak{V})$ there are finite sets K, L of variables such that $\mathfrak{V}(K, L) \cap \mathfrak{V}(B(\psi)) = \emptyset$. Let t be a member of T, $n_0, \ldots, n_k$ distinct natural numbers. By (ii) there is for every $t \in T$ a $t' > t$ such that $S_{t'} \in S(K(n_0, \ldots, n_k), L(n_0, \ldots, n_k))$. Therefore $\psi(n_0, \ldots, n_k)$ is not true at t. Thus for each choice of distinct $n_0, \ldots, n_k$ and each $t \in T$ $\varphi(n_0, \ldots, n_k)$ is false at t. Thus we may put $B(\varphi) = \{\langle 1, \langle \{v_0\}, \{v_0\} \rangle \rangle\}$.

The above argument yields the following

THEOREM 7. *Let Q be a set of predicate letters and individual constants which contains at least one predicate letter, and let $\mathfrak{K}$ be a class of partial orderings which contains at least one ordering which is isomorphic with the integers. Then there are sentences ψ of $\mathfrak{L}_{1,Q}(\mathsf{N})$ such that for no sentence φ of $\mathfrak{L}_{1,Q}$ is $\psi \leftrightarrow \varphi$ $\mathfrak{K}$-valid.*

References

[1] R. A. BULL. "An algebraic study of tense logics with linear time." *The journal of symbolic logic*, vol. 33 (1968), pp. 27—38.
[2] A. N. PRIOR. ""Now."" *Noûs*, vol. 2 (1968), pp. 101—119.
[3] D. KALISH AND R. MONTAGUE. *Logic. Techniques of formal reasoning.* New York: Harcourt, Brace & World, 1964.
[4] J. A. W. KAMP. "Tense logic and the theory of linear order." Ph. D. dissertation, U. C. L. A., 1968.
[5] E. J. LEMMON AND D. S. SCOTT. "Intensional logic. Preliminary draft of initial chapters by E. J. Lemmon." Dittoed. Stanford, Calif.: Stanford University, 1966.

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